Random Matrix Theory: essentials

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First models

Let X_n random matrix, $X_n(i,j)$ i.i.d. $\sim N(0,1)$.

- Wishart (1928): $W_n = p_1(X_n) = X_n X_n^T$
- Wigner (1955): $M_n = p_2(X_n) = \frac{X_n + X_n^T}{2}$

$$M_n \begin{cases} GOE & X_n(i,j) \in \mathbb{R}, \beta = 1 \\ GUE & X_n(i,j) \in \mathbb{C}, \beta = 2 \\ GSE & quaternions, \beta = 4 \end{cases}$$

Wigner hypothesis: the eigenvalues $\lambda_i^{(n)}$ of M_n is a good model for the energy levels of an atom.

Questions: distributions of

- $d^{(n)} = \lambda_{i+1}^{(n)} - \lambda_i^{(n)}$,
- $F_n(x) = \frac{\{\# \ i : \lambda_i^{(n)} \leq x\}}{n}$,
- $\lambda_1^{(n)}$

Wigner (1955)

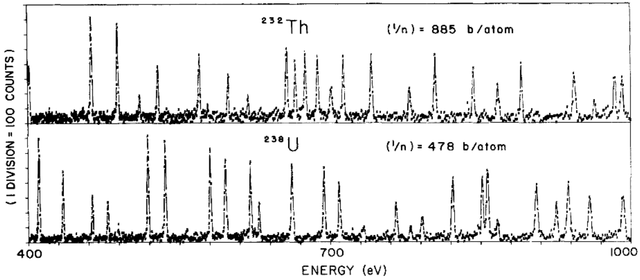

FIG. 3. Examples of the 33-m M-R data, counts versus energy from 500 to 700 eV for ^{232}Th and ^{238}U. Background has been subtracted in these plots.

From Mehta (2004)

Spacing between nuclear resonance energy

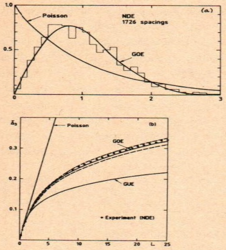

Fig.II.7 (a) Nearest-neighbour spacing histogram for NDE (experiment) ; (b) $\bar{\Delta}_3$ as a function of L ; dashed lines for GOE, to take into account finite sample size effects, correspond to one standard deviation from the average (taken from [HPB-82,BHP-83])

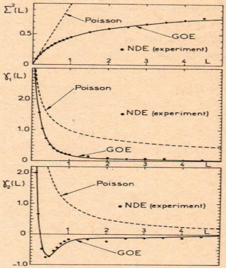

Fig.II.8 – (a) Σ^2, (b) γ_1, (c) γ_2 as functions of L (taken from [BHP-84])

From Bohigas and Giannoni (1984)

Zeros of the $\zeta-$Riemann function

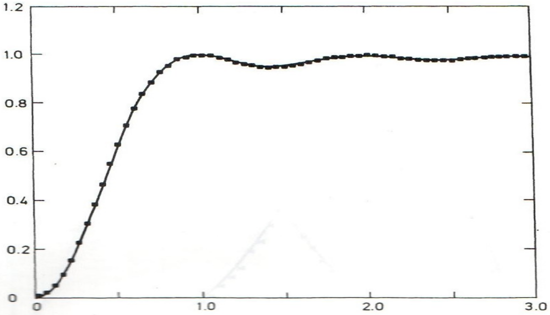

Figure 1.11. The same as Figure 1.9, but for 79 million zeros around $n \approx 10^{20}$. From Odlyzko (1989). Copyright © 1989 American Telephone and Telegraph Company, reprinted with permission.

From Metha (2004)

2-points correlation function: $1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$

Sinai's billiards

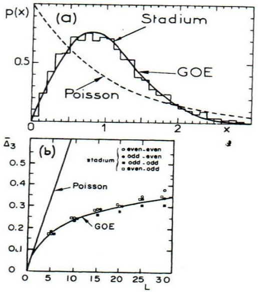

Fig. 2. — Results of level fluctuations for the eigenvalues of the stadium : (a) nearest-neighbour spacing distribution ; (b) $\bar{\Delta}_3$ as a function of L. In (b) results are obtained from the stretch of levels going from the 50th to 800th level for each symmetry class. In (a), to improve the statistics, all spacings corresponding to each symmetry are included. GOE and Poisson predictions are drawn for comparison.

Conjecture of Bohigas, Giannoni, Schmit (1984): Spectra of certain chaotic systems may behave as that of GOE.

From $\{1, 2, \ldots, n\}$ take all possible permutations P_n ($|P_n| = n!$).
e.g. 4,1,5,2,3
Each permutation $\sigma_n \in P_n$ has a longest $L(\sigma_n)$ increasing subsequence. Baik, Deif and Johanson (1999) showed after some rescaling that

$$l_n(k) = \frac{\{\#\sigma_n \in P_n : L(\sigma_n) \leq k\}}{n!} = Prob(L(\sigma_n) \leq k) \to F_2,$$

where F_2 is the so-called Tracy-Widom distribution coming from the GUE.

Proof. Using the Gessel formula

$$\sum_{n=0}^{\infty} \frac{e^{-\alpha}\alpha^n}{n!} l_n(k) = e^{-\alpha} D_k(e^{2\sqrt{\alpha}\cos(t)}),$$

where $D_k(f(t))$ is the Toeplitz determinant of order k of $f \in L^1(\partial\mathbb{D})$. And the speepest descent methods in a Riemann-Hilbert problem.

Wigner semicircle law (1955)

Limit of the Empirical Spectral Distribution (ESD) of M_n:

$$\mu_n(\bullet) = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i/\sqrt{n}}(\bullet) \to \rho_{sc}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

Proof.

$$\int_{\mathbb{R}} x^k d\mu_n = \frac{1}{n} traza\left[\left(\frac{1}{\sqrt{n}} M_n\right)^k\right] \to C_{k/2} = moments(\rho_{sc})$$

$$C_{k/2} = \begin{cases} 0 & k = 1, 3, 5, \ldots \\ \frac{k!}{(k/2+1)!(k/2)!} & k = 2, 4, \ldots \end{cases}$$

Dyck paths: # number of trayectories of a random walk that finishes at 0.

ref. Dominguez y Rocha, Miscelanea Matematica (2011).

Marchenko-Pastur law (1967)

Limit of the ESD of W_n:

$$\mu_n(\bullet) = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i/n}(\bullet) \to \sigma_{MP}(x) = \frac{1}{2\pi x}\sqrt{(1-x)(x-1)}$$

Proof. Using the Stieltjes transform (ST).

$$ST(\mu_n) = \int_{\mathbb{R}} \frac{1}{x-z} d\mu_n = \frac{1}{n} traza \left[\left(\frac{1}{\sqrt{n}} W_n - zI \right)^{-1} \right] \to ST(\sigma_{MP})$$

Circular law

ESD of X_n with independent entries.

Ginibre (1960): if $X_n(i,j)$ are Gaussian, $\mu_n \to$ *uniform on* $\mathbb{D}$.

With weaker condition: Girko (1967),. . . , Tao and Vu (2010)

$$\left(\frac{1}{\sqrt{n}}X_n - zI\right)\left(\frac{1}{\sqrt{n}}X_n - zI\right)^*$$

Distribution of eigenvalues and Logarithmic Potential

Hsu (1939). Distribution of $(\lambda_1, \ldots, \lambda_n)$

$$f(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \prod_{k=1}^{n} e^{-\frac{\beta}{4}\lambda_k^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta$$

Dyson (1962). Logarithmic potential in the Stieltjes electrostatic model

$$f(\lambda_1, \ldots, \lambda_n) = \frac{1}{Z_n} \exp\left[-\frac{\beta}{4} \sum_{k=1}^{n} V(\lambda_k) + \beta \sum_{i<j} \log|\lambda_i - \lambda_j| \right],$$

with potential $V(x) = \frac{\beta}{4} x^2$.
It turns out that the points of maximum likelihood of f are zeros of orthogonal polynomials generated by $e^{-V(x)}$.

Determinental structure. Dyson, Gaudin, Metha 60'

Using the Van der Monde determinant $\prod_{i<j} |\lambda_i - \lambda_j| =$

$$det \begin{bmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix} = det \begin{bmatrix} p_0(\lambda_1) & \dots & p_0(\lambda_n) \\ p_1(\lambda_1) & \dots & p_1(\lambda_n) \\ \vdots & \vdots & \vdots \\ p_{n-1}(\lambda_1^{n-1}) & \dots & p_{n-1}(\lambda_n^{n-1}) \end{bmatrix},$$

for polynomials $p_0, p_1, \dots$.

It turns out that

$$e^{-\sum_{k=1}^n V(\lambda_k)} \prod_{i<j} |\lambda_i - \lambda_j|^2 = \det[K_n(\lambda_i, \lambda_j)]_{i,j=1}^n,$$

where $K_n(x, y) = \sum_{k=0}^{n-1} \phi_i(x)\phi_j(y)$, $\phi_i(x) = e^{-\frac{V(x)}{2}} p_i(x)$.
Taking p_i the Hermite polynomials, and after special scaling, one obtains (see Deift (1999,2009))

$$K_n(x, y) \to \frac{\sin(x - y)}{x - y}$$

The Tracy-Widom distribution

Tracy & Widom (**1994**). Take the largest eigenvalue λ_{max},

$$\frac{\lambda_{max} - \sqrt{2n}}{(8n)^{-1/6}} \xrightarrow{d} F_2 := e^{-\int_t^\infty (x-t)q^2(x)dx}$$

where q is solution of the Painlevé equation

$$q'' = tq + 2q^3, \ \lim_{t\to\infty} q(t) = Ai(t).$$

Proofs: When working at the *edge* of the spectrum one obtains

$$\frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x-y}$$

Free probability, Dan Voicolescu 1986

• These limit laws are spectral laws of elements in an algebra. Tao(2012)

Def. A **non-commutative probability space** is a C^* algebra $\mathcal{A}$ endowed with a state $\tau : \mathcal{A} \to \mathbb{C}$ such that

$$\tau(XX^*) \geq 0 \text{ and } \tau(I) = 1$$

Example. $\mathcal{A} = \{n \times n \textit{ random complex matrices}\}$ and $\tau(M_n) = \frac{1}{n} E[traza(M_n)]$

Thm. Elements of $X \in \mathcal{A}$ adjoint has a measure μ such that

$$\tau(p(X)) = \int p(x)\mu(dx)$$

•. The concept of independent is similar but with different consequences: X and Y are free independent if

$$\tau(p_1(X)p_2(Y)p_3(X)p_4(Y)) = \tau(p_1(X))\tau(p_2(Y))\tau(p_3(X))\tau(p_4(Y))$$

F_2 & Gaussian inside the KPZ equation (interaction of particles) Corwin (2012)