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Asymptotically Good Pseudomodes for Toeplitz Matrices and Wiener-Hopf Operators

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To the memory of Erhard Meister

Abstract. We describe the structure of asymptotically good pseudomodes for Toeplitz matrices and their circulant analogues as well as for Wiener-Hopf integral operators and a continuous analogue of banded circulant matrices. The pseudomodes of circulant matrices and their continuous analogues are extended, while those of Toeplitz matrices or Wiener-Hopf operators are typically strongly localized in the endpoints.

1. Introduction

Let A be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. A point λ in C is said to be an E-pseudoeigenvalue of A if $\|(A - \lambda I)^{-1}\| \geq 1/\varepsilon$ (with the convention that $\|(A - \lambda I)^{-1}\| := \infty$ in case A – X I is not invertible). If λ is an ε-pseudoeigenvalue, then there exists a nonzero x $\in \mathcal{H}$ such that $\|(A-\lambda I)x\| \leq \varepsilon\|x\|$. Each such x is called an *ε-pseudomode* (or E-pseudoeigenvector) for A at λ. Papers [10], [12], [13], the web site [5], and the book [4] contain detailed information about these concepts.

Now suppose we are given a sequence $\{A_n\}_{n=1}^{\infty}$ of matrices A„ $\in$ C$^{n\times n}$. We think of A,, as an operator on C^n with the ℓ^2 norm. We call a point $\lambda \in \mathbf{C}$ an asymptotically good pseudoeigenvalue for {A,,) if $\|(A_n - \lambda I)^{-1}\| \to \infty$ as n $\to \infty$. In that case we can find nonzero vectors x„ $\in$ C^n satisfying

$$\|(A_n - \lambda I)x_n\|/\|x_n\| \to 0 \ \text{as} \ n \to \infty,$$

and each sequence {x,,) with this property will be called an *asymptotically* good pseudomode for $\{A_n\}$ at λ. Our terminology is motivated by the papers [10] and [14]: there it is shown that $\|$(A,,$-\lambda I)^{-1}\|$ increases exponentially for certain classes of matrices, and the corresponding pseudomodes are called *exponentially* good.

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This paper is devoted to the structure of asymptotically good pseudomodes for sequences of Toeplitz matrices. We also embark on Wiener-Hopf integral operators and on the circulant cousins of Toeplitz band matrices (called α-matrices in theoretical chemistry [16]) and their continual analogues.

2. Banded circulant matrices

Given a subset J_n of $\{1, 2, \ldots, n)$, we denote by *Pj,,* the projection on C^n defined by

$$(P_{J_n}y)_j = \begin{cases} y_j & \text{for} \quad j \in J_n, \\ 0 & \text{for} \quad j \notin J_n. \end{cases}$$

The number of elements in J_n will be denoted by $|J_n|$. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of nonzero vectors *y„* $\in C^n$. We say that *{y,,)* is *asymptotically localized* if there exists a sequence $\{J_n\}_{n=1}^{\infty}$ of sets $J_n \subset \{1, \ldots, n)$ such that

$$\lim_{n—03} \frac{|J_n|}{n} = 0 \quad \text{and} \quad \lim_{n—03} \frac{\|P_{J_n}y_n\|}{\|y_n\|} = 1.$$

We denote by $F_n \in C^{n \times n}$ the Fourier matrix,

$$F_n = \frac{1}{\sqrt{n}} \left(\omega_n^{jk}\right)_{j,k=0}^{n-1}, \quad \omega_n := e^{2\pi i/n}.$$

A sequence $\{y_n\}_{n=1}^{\infty}$ of nonzero vectors $y_n \in C^n$ will be called *asymptotically extended* if $\{F_n y_n\}$ is asymptotically localized. Problems concerning the question whether eigenvectors or pseudoeigenvectors are localized or extended have been extensively studied for many decades, especially for randomly perturbed Toeplitz matrices and their differential operators analogues, and the literature on this topic is vast. A few exemplary works are [1], [7], [8], [9], [11], [14].

Let a be a complex-valued L^∞ function on the complex unit circle T. The $n \times n$ Toeplitz matrix $T_n(a)$ and the infinite Toeplitz matrix $T(a)$ are defined by $T_n(a) = (a_{j-k})_{j,k=1}^{n}$ and $T(a) = (a_{j-k})_{j,k=1}^{\infty}$, where

$$a_\ell = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-i\ell\theta} d\theta, \quad \ell \in \mathbf{Z}.$$

As $a \in L^m(T)$, the matrix $T(a)$ generates a bounded linear operator on $\ell^2(\mathbf{N})$. Now suppose that a is actually a trigonometric polynomial,

$$a(t) = \sum_{\ell=-r}^{s} a_\ell t^\ell, \quad t \in \mathbf{T}.$$

Then $T_n(a)$ is a banded matrix. For n large enough; we can add entries in the lower-left and upper-right corners of $T_n(a)$ in order to get a circulant matrix $C_n(a)$. For example, if $a(t) = a_{-1}t^{-1} + a_0 + a_1 t + a_2 t^2$, then

$$C_6(a) = \begin{pmatrix} a_0 & a_{-1} & 0 & 0 & a_2 & a_1 \\ a_1 & a_0 & a_{-1} & 0 & 0 & a_2 \\ a_2 & a_1 & a_0 & a_{-1} & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & a_{-1} & 0 \\ 0 & 0 & a_2 & a_1 & a_0 & a_{-1} \\ a_{-1} & 0 & 0 & a_2 & a_1 & a_0 \end{pmatrix}$$

Theorem 2.1. *Let a be n trigonometric polynomial. A point $\lambda \in \mathbf{C}$ is an asymptotically good pseudoeigenvnlue for $\{C_n(a)\}$ if and only if $\lambda \in a(\mathbf{T})$, in which case every asymptotically good pseudomode for $\{C_n(a)\}$ is asymptotically extended.*

Proof. Clearly, $C_n(a) - XI = C_n(a - \lambda)$. It is well known that

$$C_n(a - \lambda) = F_n^* \operatorname{diag}\left(a(\omega_n^j) - \lambda\right)_{j=0}^{n-1} F_n =: F_n^* D, F_n.$$

Since F_n is unitary, it follows that

$$\|C_n^{-1}(a - \lambda)\| = \frac{1}{\min\limits_{0 \le j \le n-1} |a(\omega_n^j) - \lambda|},$$

which shows that $\|C_n^{-1}(a - \lambda)\| \to \infty$ if and only if $\lambda \in a(\mathbf{T})$.

Now pick $\lambda \in a(\mathbf{T})$ and suppose $\{x,)$ is an asymptotically good pseudomode for $\{C_n(a)\}$ at λ. We may without loss of generality assume that $\|x_n\| = 1$. Put $y, = (y_j^{(n)})_{j=1}^n = F_n x_n$. We have

$$\|C_n(a \quad \lambda)x_n\| = \|F_n^* D_n F_n x_n\| = \|D_n y_n\|. \tag{2.1}$$

Fix an $\varepsilon > 0$. For $6 > 0$, we put

$$\begin{aligned} G_n(\delta) &= \{j \in \{1, \ldots, n\} : |a(\omega_n^{j-1}) - \lambda| \le \delta\}, \\ E(6) &= \{\theta \in [0, 2\pi) : |a(e^{i\theta}) - \lambda| < 6). \end{aligned}$$

Since a is analytic in $\mathbf{C} \setminus \{0\}$, the set $E(\delta)$ is a finite union of intervals. Hence $|G_n(\delta)|/n \to |E(\delta)|/(2\pi)$ as $n \to \infty$, where $|E(\delta)|$ denotes the (length) measure of $E(\delta)$. Because $|E(\delta)| \to 0$ as $6 \to 0$, there exist $\delta(\varepsilon) > 0$ and $N_1(\varepsilon) \ge 1$ such that $|G_n(\delta(\varepsilon))|/n < \varepsilon$ for all $n \ge N_1(\varepsilon)$. From *(2.1)* we infer that $\|D_n y_n\|^2 \to 0$ as n $\to \infty$. Consequently, $\|D_n y_n\|^2 < \varepsilon\delta(\varepsilon)^2$ for all $n \ge N_2(\varepsilon)$. Since

$$\|D_n y_n\|^2 = \sum_{j=1}^n |a(\omega_n^{j-1}) - \lambda|^2 |y_j^{(n)}|^2 \ge \delta(\varepsilon)^2 \sum_{j \notin G_n(\delta(\varepsilon))} |y_j^{(n)}|^2,$$

it follows that $\sum\limits_{j \notin G_n(\delta(\varepsilon))} |y_j^{(n)}|^2 < \varepsilon$ for $n \ge N_2(\varepsilon)$. Thus, $\|P_{G_n(\delta(\varepsilon))} y_n\|^2 > 1 - \varepsilon$ for all $n \ge N_2(\varepsilon)$. Put $n(\varepsilon) = \max(N_{1(E)}, N_2(\varepsilon))$.

Now let $\varepsilon_k = 1/k$ $(k \ge 2)$. With $\delta_k := \delta(\varepsilon_k)$ and $n_k := n(\varepsilon_k)$ we then have

$$\frac{|G_n(\delta_k)|}{n} < \frac{1}{k} \quad \text{and} \quad \|P_{G_n(\delta_k)} y_n\|^2 > 1 - \frac{1}{k} \quad \text{for } n \ge n_k. \tag{2.2}$$

We may without loss of generality assume that $1 < n_2 < n_3 < \ldots$. For $1 \le n < n_2$, we let J_n denote an arbitrary subset of $\{1,\ldots,n\}$. For $n \ge n_2$, we define the sets $J_n \subset \{1,\ldots,n\}$ by

$$\begin{aligned} J_{n_2} &= G_{n_2}(\delta_2), J_{n_2+1} = G_{n_2+1}(\delta_2), \ldots, J_{n_3-1} = G_{n_3-1}(\delta_2), \\ J_{n_3} &= G_{n_3}(\delta_3), J_{n_3+1} = G_{n_3+1}(\delta_3), \ldots, J_{n_4-1} = G_{n_4-1}(\delta_3), \quad \ldots. \end{aligned}$$

From (2.2) we see that

$$\begin{aligned} &\frac{|J_{n_2}|}{n_2} < \frac{1}{2}, \ldots, \frac{|J_{n_3-1}|}{n_3-1} < \frac{1}{2}, \\ &\frac{|J_{n_3}|}{n_3} < \frac{1}{3}, \ldots, \frac{|J_{n_4-1}|}{n_4-1} < \frac{1}{3}, \quad \ldots, \end{aligned}$$

which shows that $|J_n|/n \to 0$ as $n \to \infty$. Also by (2.2),

$$\begin{aligned} &\|P_{J_{n_2}} y_{n_2}\|^2 > 1 - \frac{1}{2}, \ldots, \|P_{J_{n_3-1}} y_{n_3-1}\|^2 > 1 - \frac{1}{2}, \\ &\|P_{J_{n_3}} y_{n_3}\|^2 > 1 - \frac{1}{3}, \ldots, \|P_{J_{n_4-1}} y_{n_4-1}\|^2 > 1 - \frac{1}{3}, \quad \ldots, \end{aligned}$$

and hence $\|P_{J_n} y_n\| \to 1$ as $n \to \infty$. Since $\|y_n\| = 1$ for all n, it results that $\{y_n\}$ is asymptotically localized. Consequently, $\{x_n\}$ is asymptotically extended. $\square$

3. Toeplitz matrices

Let $a \in L^\infty(\mathbf{T})$ be a piecewise continuous function, that is, suppose the one-sided limits $a(t-0)$ and $a(t+0)$ exist for each t on the counter-clockwise oriented unit circle. We denote by $a^\#(\mathbf{T})$ the closed and continuous curve that results from the (essential) range of a by filling in the line segments $[a(t-0), a(t+0)]$ at each jump of a. The counter-clockwise orientation of $\mathbf{T}$ induces an orientation of $a^\#(\mathbf{T})$ in the natural manner. For $\lambda \in \mathbf{C} \setminus a^\#(\mathbf{T})$, we let wind (a,λ) denote the winding number of the curve $a^\#(\mathbf{T})$ about λ. It is well known that the spectrum of $T(a)$ on $\ell^2(\mathbf{N})$ is the union of $a^\#(\mathbf{T})$ and all points $\lambda \in \mathbf{C} \setminus a^\#(\mathbf{T})$ with wind $(a,\lambda) \ne 0$. If $\lambda \notin a^\#(\mathbf{T})$ and wind $(a,\lambda) = -m < 0$, then the kernel (= null space) $\operatorname{Ker} T(a-\lambda)$ has the dimension m, while if $\lambda \notin a^\#(\mathbf{T})$ and wind $(a,\lambda) = m > 0$, then the kernel of the adjoint of $T(a-\lambda)$ is m-dimensional. All these facts can be found in [4] or [6], for example.

Suppose that $\lambda \notin a^\#(\mathbf{T})$ and wind $(a,\lambda) = -m < 0$. We then can write $a - \lambda = b\chi_{-m}$, where b is piecewise continuous, $0 \notin b^\#(\mathbf{T})$, wind $(b,0) = 0$, and χ_k is defined by $\chi_k(t) = t^k$ $(t \in \mathbf{T})$. The operator $T(b)$ is invertible on $\ell^2(\mathbf{N})$ and, moreover, the matrices $T_n(b)$ are invertible for all sufficiently large n,

$$\lim_{n\to\infty} \|T_n^{-1}(b)\| = \|T^{-1}(b)\| \quad \text{and} \quad T_n^{-1}(b)P_n \to T^{-1}(b) \text{ strongly} \tag{3.1}$$

(see, e.g., [2], [4], [6]). Here P_n is the projection on $\ell^2(\mathbf{N})$ given by $(P_n y)_j = y_j$ for $1 \le j \le n$ and $(P_n y)_j = 0$ for $j \ge n+1$. We will frequently identify the image of

P_n with $\mathbf{C}^n$. It is also well known (and easily verified) that the m elements

$$u_j := T^{-1}(b)e_j \quad (j = 1, \dots, m) \tag{3.2}$$

form a basis in $\operatorname{Ker} T(a-\lambda)$, where $e_j \in \ell^2(\mathbf{N})$ is the sequence whose jth term is 1 and the remaining terms of which are zero. We finally remark that, obviously, $T(a) - \lambda I = T(a-\lambda)$ and $T_n(a) - \lambda I = T_n(a-\lambda)$.

Each point $\lambda \in \mathbf{C} \setminus a^{\#}(\mathbf{T})$ with $\operatorname{wind}(a, \lambda) \neq 0$ is an asymptotically good pseudoeigenvalue for $\{T_n(a)\}$ (see [4], [6], [10]). The following theorem provides us with a complete description of the structure of asymptotically good pseudomodes.

Theorem 3.1. Suppose $\lambda \notin a^{\#}(\mathbf{T})$ and $\operatorname{wind}(a, \lambda) = -m < 0$. Let $x_n \in \mathbf{C}^n$ be unit vectors. The sequence $\{x_n\}$ is an asymptotically good pseudomode for $\{T_n(a)\}$ at λ if and only if there exist $c_1^{(n)}, \dots, c_m^{(n)} \in \mathbf{C}$ and $z_n \in \mathbf{C}^n$ such that

$$x_n = c_1^{(n)} P_n u_1 + \cdots + c_m^{(n)} P_n u_m + z_n, \tag{3.3}$$

$$\sup_{n \geq 1,\, 1 \leq j \leq m} |c_j^{(n)}| < \infty, \qquad \lim_{n \to \infty} \|z_n\| = 0, \tag{3.4}$$

where $u_1, \dots, u_m$ are given by (3.2).

Proof. Assume that (3.3) and (3.4) hold. Since $\|T_n(a-\lambda)P_n\| \leq \|a - \lambda\|_\infty$, we see that $T_n(a-\lambda)z_n \to 0$. As the numbers $|c_j^{(n)}|$ are bounded by a constant independent of n and as $P_n \to \mathbf{I}$ strongly (= pointwise) and $T(a-\lambda)u_j = 0$, we obtain that

$$\lim_{n\to\infty} T_n(a-\lambda)x_n = \sum_{j=1}^{m} \lim_{n\to\infty} c_j^{(n)} T_n(a-\lambda)u_j = 0.$$

Thus, $\{x_n\}$ is an asymptotically good pseudomode.

Conversely, suppose $\|T_n(a-\lambda)x_n\| \to 0$. Put $y_n = T_n(a-\lambda)x_n$. With $Q_n = \mathbf{I} - P_n$, we have

$$\begin{aligned} T_n(a-\lambda) &= T_n(\chi_{-m}b) = P_n T(\chi_{-m}b) P_n = P_n T(\chi_{-m}) T(b) P_n \\ &= P_n T(\chi_{-m}) P_n T(b) P_n + P_n T(\chi_{-m}) Q_n T(b) P_n =: A_n + B_n. \end{aligned}$$

Since $T(\chi_{-m})$ is nothing but the shift operator $(\xi_1, \xi_2, \dots) \mapsto (\xi_{m+1}, \xi_{m+2}, \dots)$, it follows that

$$\operatorname{Im} A_n \subset \operatorname{Im} P_{n-m}, \quad \operatorname{Im} B_n \subset \operatorname{Im} P_{\{n-m+1, \dots, n\}}, \tag{3.5}$$

where $\operatorname{Im} C$ refers to the image (= range) of the operator C. This implies that

$$\|y_n\|^2 = \|A_n x_n + B_n x_n\|^2 = \|A_n x_n\|^2 + \|B_n x_n\|^2,$$

and hence $\|A_n x_n\| \to 0$ because $\|y_n\| \to 0$. The equality $T_n(\chi_{-m})T_n(b)x_n = A_n x_n$ gives

$$T_n(b)x_n = c_1^{(n)} e_1 + \ldots + c_m^{(n)} e_m + T_n(\chi_m) A_n x_n$$

with certain complex numbers $c_1^{(n)}, \ldots, c_m^{(n)}$. Since

$$\left(\sum_{j=1}^{m} |c_j^{(n)}|^2\right)^{1/2} \le \|T_n(b)x_n\| + \|T(\chi_m)\| \, \|A_n x_n\| \le \|b\|_\infty + \|A_n x_n\|,$$

we conclude that there is an $M < \infty$ such that $|c_j^{(n)}| \le M$ for all n and j. Finally, from (3.1), (3.2) and the equality

$$x_n = c_1^{(n)} T_n^{-1}(b) e_1 + \ldots + c_m^{(n)} T_n^{-1}(b) e_m + T_n^{-1}(b) T_n(\chi_m) A_n x_n$$

we get (3.3) and (3.4) with

$$z_n = T_n^{-1}(b) T_n(\chi_m) A_n x_n + \sum_{j=1}^{m} c_j^{(n)} (T_n^{-1}(b) e_j - P_n T^{-1}(b) e_j).$$

This completes the proof. □

Let a be as in Theorem 3.1, but in addition suppose now that a belongs to the Wiener algebra $W(\mathbf{T})$, i.e., that the Fourier series of a converges absolutely. We write $a - \lambda = b\chi_{-m}$ as above. Clearly, b is also in $W(\mathbf{T})$. Since $0 \notin b(\mathbf{T})$ and $\operatorname{wind}(b, 0) = 0$, the function b has a Wiener-Hopf factorization $b = b_- b_+$. The factors $b_\pm$ can be given by

$$b_-(t) = \exp\left(\sum_{\ell=0}^{\infty} (\log b)_{-\ell} t^{-\ell}\right), \quad b_+(t) = \exp\left(\sum_{\ell=1}^{\infty} (\log b)_\ell t^\ell\right),$$

where $\log b$ is any logarithm of b in $W(\mathbf{T})$. The Wiener-Hopf factorization $b = b_- b_+$ yields the representation $T^{-1}(b) = T(b_+^{-1}) T(b_-^{-1})$ (see, e.g., [4] or [6]), or written down in detail, $T^{-1}(b)$ equals

$$\begin{pmatrix} (b_+^{-1})_0 & & & \\ (b_+^{-1})_1 & (b_+^{-1})_0 & & \\ (b_+^{-1})_2 & (b_+^{-1})_1 & (b_+^{-1})_0 & \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix} \begin{pmatrix} (b_-^{-1})_0 & (b_-^{-1})_{-1} & (b_-^{-1})_{-2} & \ldots \\ & (b_-^{-1})_0 & (b_-^{-1})_{-1} & \ldots \\ & & (b_-^{-1})_0 & \ldots \\ & & & \ldots \end{pmatrix}$$

If a is even rational, then the sequences $\{(b_+^{-1})_n\}_{n=0}^\infty$ and $\{(b_-^{-1})_{-n}\}_{n=0}^\infty$ decay exponentially, and from (3.2) we deduce that $u_1, \ldots, u_m$ are also exponentially decaying. Thus, Theorem 3.1 implies that, up to the $o(1)$ term z_n, all asymptotically good pseudomodes are exponentially decaying. We remark that in the case where a is a trigonometric polynomial (which is equivalent to the requirement that $T(a)$ be a banded matrix) the existence of exponentially decaying pseudomodes was already proved in [10] and [14].

We now sharpen the definition of an asymptotically localized sequence. We say that a sequence $\{y_n\}$ of vectors $y_n \in \mathbf{C}^n$ is asymptotically strongly localized in the beginning part if

$$\lim_{n\to\infty} \frac{\|P_{\{1,\ldots,j_n\}} y_n\|}{\|y_n\|} = 1 \tag{3.6}$$

for every sequence $\{j_n\}_{n=1}^{\infty}$ such that $j_n \to \infty$ and $1 \le j_n \le n$. Asymptotic strong localization in the beginning part implies, for example, that *(3.6)* is true with $j_n = \log\log n$ $(n \ge 3)$.

Theorem 3.2. *Suppose* $\lambda \notin a^{\#}(\mathbf{T})$ *and* wind$(a,\lambda) = -m < 0$. *Then every asymptotically good pseudomode for {?;,(a)) at* λ *is asymptotically strongly localized in the beginning part.*

Proof. Let *{z,,)* be an asymptotically good pseudomode for $\{T_n(a)\}$ at λ. We may without loss of generality assume that $\|x_n\| = 1$ for all n. By Theorem *3.1*,

$$x_n = x_1^{(n)} P_n u_1 + \cdots + c_m^{(n)} P_n u_m + z_n =: w_n + z_n,$$

where $u_1, \ldots, u_m$ are given by *(3.2)* and $c_1^{(n)}, \ldots, c_m^{(n)}, z_n$ satisfy *(3.4)*. Choose $M < \infty$ so that $|c_i^{(n)}| \le M$ for all i and n. Let $\{j_n\}$ be any sequence such that $j_n \to \infty$ and $1 \le j_n \le n$. Put $J_n = \{1, \ldots, j_n\}$ and $J_n^c = \{j_n + 1, \ldots, n\}$. From *(3.2)* we infer that $u_1, \ldots, u_m \in \ell^2(\mathbf{N})$. We have $\|P_{J_n^c} w_n\| \le M \sum_{i=1}^{m} \|P_{J_n^c} u_i\|$. Since $u_i = (u_k^{(i)})_{k=1}^{\infty}$ is in $\ell^2(\mathbf{N})$ and hence

$$\|P_{J_n^c} u_i\|^2 = \sum_{k=j_n+1}^{n} |u_k^{(i)}|^2 \le \sum_{k=j_n+1}^{\infty} |u_k^{(i)}|^2 = o(1) \quad \text{as} \quad j_n \to \infty,$$

it follows that $\|P_{J_n^c} w_n\| \to 0$ as $n \to \infty$. Finally,

$$\begin{aligned} 1 \ge \|P_{J_n} x_n\|^2 &= 1 - \|P_{J_n^c} x_n\|^2 = 1 - \|P_{J_n^c}(w_n + z_n)\|^2 \\ &\ge 1 - \Big(\|P_{J_n^c} w_n\| + \|P_{J_n^c} z_n\|\Big)^2, \end{aligned}$$

and because $\|P_{J_n^c} w_n\| \to 0$ and $\|P_{J_n^c} z_n\| \to 0$ as $n \to \infty$, we arrive at the conclusion that $\|P_{J_n} x_n\| \to 1$. $\square$

To conclude this section, suppose that $\lambda \in \mathbf{C} \setminus a^{\#}(\mathbf{T})$ and that wind$(a,\lambda) = m > 0$. Following [15], we define $\widetilde{a}$ by $\widetilde{a}(t) := a(1/t)$ $(t \in \mathbf{T})$ and we let W_n be the operator that is P_n followed by reversal of the coordinates. We have $\lambda \notin \widetilde{a}^{\#}(\mathbf{T})$ and wind$(\widetilde{a}, \lambda) = -m < 0$. Moreover, $W_n T_n(a - \lambda) W_n = T_n(\widetilde{a} - \lambda)$ and hence $\|T_n(a - \lambda)x_n\| = \|T_n(\widetilde{a} - \lambda) W_n x_n\|$. Consequently, by Theorem *3.1*, a sequence $\{x_n\}$ of unit vectors is an asymptotically good pseudomode of $\{T_n(a)\}$ at λ if and only if

$$W_n x_n = c_1^{(n)} P_n \widetilde{u}_1 + \ldots + c_m^{(n)} P_n \widetilde{u}_m + z_n, \tag{3.7}$$

where $|c_j^{(n)}| \le M < \infty$ for all j and n, $\|z_n\| \to 0$ as $n \to \infty$, and $\widetilde{u}_1, \ldots, \widetilde{u}_m$ are given by $\widetilde{u}_j = T^{-1}(\widetilde{b}) e_j$. Clearly, *(3.7)* can be rewritten in the form

$$x_n = c_1^{(n)} W_n \widetilde{u}_1 + \ldots + c_m^{(n)} W_n \widetilde{u}_m + \widetilde{z}_n$$

with $\widetilde{z}_n = W_n z_n$. The analogue of Theorem *3.2* says that every asymptotically good pseudomode $\{x_n\}$ for $\{T_n(a)\}$ at λ is asymptotically strongly localized in the terminating part, that is, the sequence $\{W_n x_n\}$ is asymptotically strongly localized in the beginning part.

4. A continuous analogue of banded circulant matrices

Let k be a function in $L^1(R)$ and suppose $k(x) = 0$ for $|x| \geq r$. For $\tau > 2r$, there is a unique continuation of k to a r-periodic function k, on all of $\boldsymbol{R}$. A continuous analogue of the operator $C_n(a)$ considered in Section 2 is the operator on $L^2(0,r)$ that is defined by

$$(C_\tau(k)f)(x) = \gamma f(x) + \int_0^\tau k_\tau(x-t)f(t)dt, \quad x \in (0,\tau),$$

where y is a fixed number in $\boldsymbol{C}$. We put

$$\widehat{k}(\xi) = \gamma + \int_{-r}^{r} k(x)e^{-i\xi x}dx, \quad \xi \in \mathbf{R}.$$

We call a point $\lambda \in \boldsymbol{C}$ an *asymptotically good pseudoeigenvalue* for $\{C_\tau(k)\}$ if $\|(C_\tau(k) - \lambda I)^{-1}\| \to \infty$ as $\tau \to \infty$, and a family $\{f_\tau\}_{\tau>0}$ of nonzero functions $f_\tau \in L^2(0,r)$ is said to be an *asymptotically good pseudomode* for $\{C_\tau(k)\}$ at λ if $\|(C_\tau(k) - \lambda I)f_\tau\|/\|f_\tau\| \to 0$ as $\tau \to \infty$.

Let $\{g_\tau\}_{\tau>0}$ be a family of elements $g, = (g_j^{(\tau)})_{j\in\mathbf{Z}} \in \ell^2(\mathbf{Z})$. We say that $\{g,\}$ is *asymptotically localized* if there exists a family $\{J_\tau\}_{\tau>\tau_0}$ of finite subsets $J, \in Z$ such that

$$\lim_{\tau\to\infty} \frac{|J_\tau|}{\tau} = 0 \quad \text{and} \quad \lim_{\tau\to\infty} \frac{\|P_{J_\tau} g_\tau\|}{\|g_\tau\|} = 1.$$

Put $\varphi_j(x) = (1/\sqrt{\tau})e^{2\pi ijx/\tau}$. The system $\{\varphi_j\}_{j\in\mathbf{Z}}$ is an orthonormal basis in $L^2(0,r)$. Thus, the map

$$\Phi_\tau : L^2(0,\tau) \to \ell^2(\mathbf{Z}), \; f_\tau \mapsto \big((f_\tau, \varphi_j)\big)_{j\in\mathbf{Z}}$$

is a unitary operator. A family $\{f_\tau\}_{\tau>0}$ of functions $f_\tau \in L^2(0,r)$ will be called *asymptotically extended* if the family $\{\Phi_\tau f_\tau\}_{\tau>0}$ is asymptotically localized.

Theorem 4.1. *A number $\lambda \in \boldsymbol{C}$ is an asymptotically good pseudoeigenvalue for $\{C_\tau(k)\}$ if and only if $\lambda \in y + \widehat{k}(\dot{\mathbf{R}})$, where $\boldsymbol{R} := \boldsymbol{R} \cup \{\infty\}$. If $\lambda + y \in \widehat{k}(\dot{\mathbf{R}})$ and, in addition, $\lambda \neq y$, then every asymptotically good pseudomode for $\{C_\tau(k)\}$ at λ is asymptotically extended.*

Proof. For $f_\tau \in L^2(0,r)$ and $x \in (0,\tau)$,

$$\begin{aligned}(C_\tau(k)f_\tau)(x) &= \gamma f(x) + \sum_{j\in\mathbf{Z}} (f_\tau, \varphi_j) \int_0^\tau k_\tau(x-t)\varphi_j(t)dt \\ &= \gamma f(x) + \sum_{j\in\mathbf{Z}} (f_\tau, \varphi_j) \frac{1}{\sqrt{\tau}} e^{2\pi ijx/\tau} \int_{x-\tau}^{x} k_\tau(s)e^{-2\pi ijs/\tau}ds\end{aligned}$$

$$
\begin{aligned}
&= \gamma f(x) + \sum_{j\in\mathbf{Z}} (f_\tau, \varphi_j) \frac{1}{\sqrt{\tau}}\, e^{2\pi i j x/\tau} \int_{-\tau/2}^{\tau/2} k_\tau(s) e^{-2\pi i j s/\tau} ds \\
&= \gamma f(x) + \sum_{j\in\mathbf{Z}} (f_\tau, \varphi_j) \frac{1}{\sqrt{\tau}}\, e^{2\pi i j x/\tau} \int_{-\tau}^{\tau} k_\tau(s) e^{-2\pi i j s/\tau} ds \\
&= \sum_{j\in\mathbf{Z}} \left(\gamma + \widehat{k}\left(\frac{2\pi j}{\tau}\right)\right) (f_\tau, \varphi_j)\varphi_j(x).
\end{aligned}
$$

Thus, $C_\tau(k)$ is unitarily equivalent to the diagonal operator

$$
\operatorname{diag}\left(\gamma + \widehat{k}\left(\frac{2\pi j}{\tau}\right)\right)_{j\in\mathbf{Z}} : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})
$$

This shows that λ is an asymptotically good pseudoeigenvalue for $\{C_\tau(k)\}$ if and only if $\lambda \in \gamma + \widehat{k}(\dot{\mathbf{R}})$.

Now suppose $\lambda = \gamma + \widehat{k}(\xi_0)$ with $\xi_0 \in \mathbf{R}$ and let $\{f_\tau\}$ be an asymptotically good pseudomode for $\{C_\tau(k)\}$ at λ. Without loss of generality we may assume that $\|f_\tau\| = 1$ for all τ. Thus,

$$
\|C_\tau(k-\lambda) f_\tau\|^2 = \sum_{j\in\mathbf{Z}} \left|\widehat{k}\left(\frac{2\pi j}{\tau}\right) - \widehat{k}(\xi_0)\right|^2 |(f_\tau, \varphi_j)|^2 \to 0. \qquad (4.1)
$$

Fix $\varepsilon > 0$. For $\delta > 0$, consider the sets

$$
\begin{aligned}
G_\tau(\delta) &= \{j \in \mathbf{Z} : |\widehat{k}(2\pi j/\tau) - \widehat{k}(\xi_0)| < \delta\}, \\
E(\delta) &= \{\xi \in \mathbf{R} : |\widehat{k}(\xi) - \widehat{k}(\xi_0)| < \delta\}.
\end{aligned}
$$

Since $\widehat{k}(\xi_0) \neq 0$ and since $\widehat{k}$ is an entire function, the set $E(\delta)$ is a finite union of intervals and $|E(\delta)| \to 0$ as $\delta \to 0$. As $|G_\tau(\delta)|/\tau \to |E(\delta)|/(2\pi)$, there are $\delta(\varepsilon) > 0$ and $t_1(\varepsilon) > 0$ such that $|G_\tau(\delta(\varepsilon))|/\tau < \varepsilon$ whenever $\tau \geq t_1(\varepsilon)$. We have

$$
\sum_{j\in\mathbf{Z}} \left|\widehat{k}\left(\frac{2\pi j}{\tau}\right) - \widehat{k}(\xi_0)\right|^2 |(f_\tau, \varphi_j)|^2 \geq \delta(\varepsilon)^2 \sum_{j\notin G_\tau(\delta(\varepsilon))} |(f_\tau, \varphi_j)|^2,
$$

which in conjunction with (4.1) gives $\|P_{G_\tau(\delta(\varepsilon))}\Phi_\tau f_\tau\|^2 > 1 - \varepsilon$ for all $\tau \geq \tau_2(\varepsilon)$. Let $\tau(\varepsilon) = \max(t_1(\varepsilon), t_2(\varepsilon))$.

Choose $\varepsilon_\ell = 1/\ell$ ($\ell \geq 2$) and put $\delta_\ell := \delta(\varepsilon_\ell)$ and $\tau_\ell := \tau(\varepsilon_\ell)$. We have proved that

$$
\frac{|G_\tau(\delta_\ell)|}{\tau} < \frac{1}{\ell} \quad \text{and} \quad \|P_{G_\tau(\delta_\ell)}\Phi_\tau f_\tau\|^2 > 1 - \frac{1}{\ell} \quad \text{for } \tau \geq \tau_\ell, \qquad (4.2)
$$

and we may assume that $0 < \tau_2 < \tau_3 < \ldots$. Let J_τ be an arbitrary subset of $\mathbf{Z}$ for $0 < \tau < \tau_2$ and define $J_\tau = G_\tau(\delta_\ell)$ for $\tau_\ell \leq \tau < \tau_{\ell+1}$. Then, by (4.2),

$$
\frac{J_\tau}{\tau} < \frac{1}{\ell} \quad \text{and} \quad \|P_{J_\tau}\Phi_\tau f_\tau\|^2 > 1 - \frac{1}{\ell} \quad \text{for } \tau_\ell \leq \tau < \tau_{\ell+1},
$$

which shows that $|J_\tau|/\tau \to 0$ and $\|P_{J_\tau}\Phi_\tau f_\tau\| \to 1$ as $\tau \to \infty$. Thus, $\{\Phi_\tau f_\tau\}$ is asymptotically localized. This means that $\{f_\tau\}$ is asymptotically extended. □

Things are different for $\lambda = y$. In the following, $P_{(-\mu\tau,\mu\tau)}$ denotes the canonical projection of $\ell^2(\mathbf{Z})$ onto the subspace of all sequences whose support is contained in the interval $(-\mu\tau, \mu\tau)$.

Theorem *4.2. Let $\{f_\tau\}_{\tau>0}$ be an asymptotically good pseudomode for $\{C_\tau(k)\}$ at γ and assume Without loss of generality that $\|f_\tau\| = 1$ for all τ. If $\widehat{k}(0) \neq \gamma$, then there exists a $\mu > 0$ such that*

$$\|P_{(-\mu\tau,\mu\tau)}\Phi_\tau f_\tau\| \to 0 \quad as \quad \tau \to \infty. \tag{4.3}$$

If $\widehat{k}(0) = \gamma$, then there are μ and ν such that $0 < \nu < \mu$ and

$$\|P_{(-\mu\tau,-\nu\tau)\cup(\nu\tau,\mu\tau)}\Phi_\tau f_\tau\| \to 0 \quad as \quad \tau \to \infty. \tag{4.4}$$

Proof. Suppose first that $\widehat{k}(0) \neq y$. Then $|\widehat{k}(\xi) - \gamma| \geq \delta > 0$ for all $|\xi| \leq 2\pi\mu$ with some $\mu > 0$. Thus, $|\widehat{k}(2\pi j/\tau) - \gamma| \geq \delta > 0$ whenever $|j| < \mu\tau$. Since

$$\begin{aligned}\|C_\tau(k-\gamma)f_\tau\|^2 &= \sum_{j\in\mathbf{Z}} \left|\widehat{k}\left(\frac{2\pi j}{\tau}\right) - \gamma\right|^2 |(f_\tau,\varphi_j)|^2 \\ &\geq \delta^2 \sum_{|j|<\mu\tau} |(f_\tau,\varphi_j)|^2 = \delta^2\|P_{(-\mu\tau,\mu\tau)}\Phi_\tau f_\tau\|^2,\end{aligned}$$

we obtain *(4.3)* from *(4.1)*. Now suppose $\widehat{k}(0) = y$. Then the entire $\widehat{k} - \gamma$ has no other zeros than 0 in some open neighborhood of 0, and hence we can find numbers μ and ν such that $0 < \nu < \mu$ and $|\widehat{k}(\xi) - \gamma| \geq \delta > 0$ for $2\pi\nu < |\xi| < 2\pi\mu$. It follows that $|\widehat{k}(2\pi j/\tau) - \gamma| \geq \delta > 0$ for $\nu\tau < |j| < \mu\tau$ and hence

$$\|C_\tau(k-\gamma)f_\tau\|^2 \geq \delta^2 \sum_{\nu\tau<|j|<\mu\tau} |(f_\tau,\varphi_j)|^2 = \delta^2\|P_{(-\mu\tau,-\nu\tau)\cup(\nu\tau,\mu,\tau)}\Phi_\tau f_\tau\|^2.$$

It is again (4.1) that implies (4.4). □

Remark 4.3. Theorem *4.2* can be improved by taking into account the orders of the zeros of $\widehat{k} - \gamma$ at the origin and at infinity. We will not embark on this question here. We rather wish to point out that Theorem *4.2* is best possible in general.

Let $k(x) = 0$ for $|x| \geq 1$ and $k(x) = 1/2$ for $|x| < 1$. Then $\widehat{k}(\xi) = \dfrac{\sin\xi}{\xi}$ and, for $f_\tau \in L^2(0,\tau)$,

$$\|C_\tau(k)f_\tau\|^2 = \sum_{j\in\mathbf{Z}} \frac{\tau^2}{4\pi^2 j^2} \sin^2\frac{2\pi j}{\tau} |(f_\tau,\varphi_j)|^2.$$

We have $\widehat{k}(0) = 1 \neq 0$. Let *{f,)* by any asymptotically good pseudomode for $\{C_\tau(k)\}$ at 0. Thus, $\|f_\tau\| = 1$ for all $\tau > 0$ and $\|C_\tau(k)f_\tau\| \to 0$ as $\tau \to$ *co*. Change f_τ to φ_τ for $\tau \in \mathrm{N}$. Since

$$\|C_\tau(k)\varphi_\tau\|^2 = \frac{\tau^2}{4\pi^2 j^2} \sin^2\frac{2\pi\tau}{\tau} = 0,$$

it follows that the family $\{h_\tau\}_{\tau>0}$ defined by h, $= f_\tau$ for $\tau \notin$ N and h, $= \varphi_\tau$ for $\tau \in$ N is also an asymptotically good pseudornode for $\{C_\tau(k)\}$ at the origin. If $\beta : (0,\infty) \to (0, \text{co})$ is any superlinear function, that is, any function that increases faster than every linear function, then

$$\|P_{(-\beta(\tau),\beta(\tau))}\Phi_\tau h_\tau\|^2 = \|P_{(-\beta(\tau),\beta(\tau))}e_\tau\|^2 = 1$$

whenever $\tau \in$ N and $\beta(\tau) > \tau$. This shows that (4.3) is not in general true with $\mu\tau$ replaced by $\beta(\tau)$.

Now let $k(x) = 0$ for $|x| \geq 1$, $k(x) = -i/2$ for $-1 <$ x $<$ 0, and $k(x) = i/2$ for $0 \leq x < 1$. Then $\widehat{k}(\xi) = (1 - \cos\xi)/\xi$. This time $\widehat{k}(0) = 0$. Since $C_\tau(k)\varphi_\tau = 0$ for $\tau \in$ N, we can argue as above to see that (4.4) is not in general valid with $\mu\tau$ and $\nu\tau$ replaced by a superlinear and sublinear function, respectively. □

5. Wiener-Hopf integral operators

The Wiener-Hopf integral operator $W(a)$ generated by a function a $\in L^\infty(\mathbf{R})$ is the bounded linear operator on $L^2(0,\text{co})$ that is defined by $W(a)$f $= PF^{-1}M(a)Ff$, where **F** is the Fourier transform,

$$(Ff)(\xi) = \int_{-\infty}^{\infty} f(x)e^{i\xi x}dx, \quad \xi \in \mathbf{R},$$

$M(a)$ stands for the operator of multiplication by a, and **P** is the orthogonal projection of L^2(R) onto $L^2(0,\infty)$. If a is of the form a $= \gamma +$ Fk with $\gamma \in \boldsymbol{C}$ and k $\in L^1$(R), then $W(a)$ can be written as

$$(W(a)f)(x) = \gamma f(x) + \int_0^\infty k(x-t)f(t)dt, \quad x > 0.$$

The Cauchy singular integral operator on the half-line,

$$(Sf)(x) = \frac{1}{\pi i}\text{ v.p.} \int_0^\infty \frac{f(t)}{t-x}\,dt, \quad x > 0,$$

is $W(\sigma)$ with the piecewise continuous function $\sigma(\xi) := -\text{sign}\,\xi$. The subject of this section is truncated Wiener-Hopf operators: for $\tau > 0$, the truncated Wiener-Hopf operator W_τ(a) is the compression of $W(a)$ to $L^2(0,\tau)$.

If a $\in L^\infty(\mathbf{R})$ is piecewise continuous, we define $a^\#(\mathbf{R})$ as the closed, continuous, and naturally oriented curve that results from the (essential) range of a by filling in the line segments $[a(x-0), a(x+0)]$ for the jumps at x $\in$ R and the line segment $[a(+\infty), a(-\infty)]$ if a has a jump at infinity. We let wind (a,λ) denote the winding number of $a^\#(\mathbf{R})$ about $\lambda \in \boldsymbol{C} \setminus a^\#(\mathbf{R})$.

Let L_j be the normalized Laguerre polynomial of degree $j-1$, The system $\{e_j\}_{j=1}^{\infty}$ given by $e_j(x) = \sqrt{2}\, L_j(2x)e^{-x}$ is an orthonormal basis in $L^2(0,\infty)$. Notice that

$$(Fe_j)(\xi) = i\sqrt{2}\left(\frac{\xi - i}{\xi + i}\right)^{j-1} \frac{1}{\xi + i}, \quad \xi \in \mathbf{R}.$$

For $k \in \mathbf{Z}$, we put

$$\chi_k(\xi) = \left(\frac{\xi - i}{\xi + i}\right)^{k}, \quad \xi \in \mathbf{R}.$$

Suppose $a \in L^\infty(\mathbf{R})$ is piecewise continuous, $\lambda \notin a^{\#}(\mathbf{R})$, and $\operatorname{wind}(a,\lambda) = -m < 0$. Then $a - \lambda = b\chi_{-m}$, where b is piecewise continuous, $0 \notin b^{\#}(\mathbf{R})$, and $\operatorname{wind}(b,\lambda) = 0$. It follows that $W(b)$ is invertible, that $W_\tau(b)$ is invertible for all sufficiently large τ, that

$$\lim_{\tau\to\infty} \|W_\tau^{-1}(b)\| = \|W^{-1}(b)\| \quad \text{and} \quad W_\tau^{-1}(b)P_\tau \to W^{-1}(b) \text{ strongly},$$

where P_τ is the natural projection of $L^2(0,\infty)$ onto $L^2(0,\tau)$, and that the functions $u_j = W^{-1}(b)e_j$ $(j = 1,\dots,m)$ form a basis in $\operatorname{Ker} W(a-\lambda)$ (see [2], [3], [6]).

Theorem 5.1. Let $\lambda \notin a^{\#}(\mathbf{R})$ and $\operatorname{wind}(a,\lambda) = -m < 0$. Let further $\{f_\tau\}_{\tau>0}$ be a family of functions $f_\tau \in L^2(0,\tau)$ of norm 1. Then the point λ is an asymptotically good pseudoeigenvalue of $\{W_\tau(a)\}_{\tau>0}$. The family $\{f_\tau\}$ is an asymptotically good pseudomode for $\{W_\tau(a)\}_{\tau>0}$ at λ if and only if there exist complex numbers $c_1^{(\tau)},\dots,c_m^{(\tau)}$ and functions $z_\tau \in L^2(0,\tau)$ such that

$$f_\tau = c_1^{(\tau)}P_\tau u_1 + \cdots + c_m^{(\tau)}P_\tau u_m + z_\tau,$$
$$\limsup_{\tau\to\infty} \max_{1\le j\le m} |c_j^{(\tau)}| < \infty, \quad \lim_{\tau\to\infty} \|z_\tau\| = 0,$$

Proof. That λ is an asymptotically good pseudoeigenvalue was established in [6] (also see [2]). The rest of the proof is analogous to the proof of Theorem 3.1, except for a modification of (3.5). Thus, suppose $W_\tau(a-\lambda)f_\tau \to 0$. Let $Q_\tau = \mathbf{I} - \mathbf{P}_\tau$. Then

$$W_\tau(a-\lambda)f_\tau = P_\tau W(\chi_{-m})P_\tau W(b)P_\tau f_\tau + P_\tau W(\chi_{-m})Q_\tau W(b)P_\tau f_\tau.$$

Put $A_\tau = P_\tau W(\chi_{-m})P_\tau W(b)P_\tau$, $B_\tau = P_\tau W(\chi_{-m})Q_\tau W(b)P_\tau$, $h_\tau = W(b)P_\tau f_\tau$. We denote by $\mathcal{P}_m$ the orthogonal projection of $L^2(0,\infty)$ onto the linear hull of $e_1,\dots,e_m$ and we set $\mathcal{Q}_m = \mathbf{I} - \mathcal{P}_m$. We have $P_\tau W(\chi_{-m})P_\tau = W(\chi_{-m})P_\tau$ and hence, $(\cdot,\cdot)$ denoting the inner product in $L^2(0,\infty)$,

$$\begin{aligned}
(A_\tau f_\tau, B_\tau f_\tau) &= (P_\tau W(\chi_{-m})P_\tau h_\tau, P_\tau W(\chi_{-m})Q_\tau h_\tau) \\
&= (W(\chi_{-m})P_\tau h_\tau, P_\tau W(\chi_{-m})Q_\tau h_\tau) \\
&= (P_\tau W(\chi_{-m})P_\tau h_\tau, W(\chi_{-m})Q_\tau h_\tau) \\
&= (W(\chi_{-m})P_\tau h_\tau, W(\chi_{-m})Q_\tau h_\tau) \\
&= (P_\tau h_\tau, W^*(\chi_{-m})W(\chi_{-m})Q_\tau h_\tau).
\end{aligned}$$

Since $W^*(\chi_{-m}) = W(\chi_m)$ and $W(\chi_m)W(\chi_{-m}) = \mathcal{Q}_m$, it follows that

$$\begin{aligned}(A_\tau f_\tau, B_\tau f_\tau) &= (P_\tau h_\tau, \mathcal{Q}_m Q_\tau h_\tau) \\ &= (P_\tau h_\tau, Q_\tau h_\tau) - (P_\tau h_\tau, \mathcal{P}_m Q_\tau h_\tau) = -(P_\tau h_\tau, \mathcal{P}_m Q_\tau h_\tau).\end{aligned}$$

Thus,

$$|(A_\tau f_\tau, B_\tau f_\tau)| \le \|P_\tau h_\tau\| \, \|\mathcal{P}_m Q_\tau\| \, \|h_\tau\| \le C\|\mathcal{P}_m Q_\tau\|$$

with some constant $C < \infty$ independent of τ. As $\mathcal{P}_m$ is compact and $Q_\tau^* =$ Q, converges strongly to zero, we may conclude that $\|\mathcal{P}_m Q_\tau\| \to 0$. Consequently, (A, f,, B, f,) $\to 0$ as $\tau \to \infty$. Finally, because $\|A_\tau f_\tau + B_\tau f_\tau\| \to 0$ and

$$\|A_\tau f_\tau + B_\tau f_\tau\|^2 = \|A_\tau f_\tau\|^2 + \|B_\tau f_\tau\|^2 + 2\,\mathrm{Re}\,(A_\tau f_\tau, B_\tau f_\tau),$$

we obtain that $\|A_\tau f_\tau\| \to 0$. The rest is as in the proof of Theorem 3.1. ☐

We call a family $\{f_\tau\}_{\tau>0}$ of nonzero functions $f_\tau \in$ L^2(0,r) asymptotically strongly localized in the beginning part if

$$\lim_{\tau\to\infty} \frac{\|P_{(0,s_\tau)} f_\tau\|}{\|f_\tau\|} = 1$$

whenever $s_\tau \to \infty$ as $\tau \to \infty$ and $0 < s_\tau < \tau$ for all τ. Again notice that s_τ is allowed to increase as slowly as desired (or required). For example, the choice $s_\tau = \log\log\log\tau$ (τ large enough) is admitted.

Theorem 5.2. **If** $\lambda \notin a^{\#}(\mathbf{R})$ and wind (a,λ) $= -m < 0$, then every asymptotically good *pseudomode* for $\{W_\tau(a)\}$ at λ is asymptotically strongly localized in the beginning part.

This can be proved by the same arguments as in the proof of Theorem **3.2.**

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