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Complex Analysis and Operator Theory

On the Structure of the C^*-Algebra Generated by Toeplitz Operators with Piece-wise Continuous Symbols

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To the memory of G. S. Litvinchuk

Abstract. We study the C^*-algebra generated by Toeplitz operators with piece-wise continuous symbols acting on the Bergman space $\mathcal{A}^2(\mathbb{D})$ on the unit disk $\mathbb{D}$ in $\mathbb{C}$. We describe explicitly each operator from this algebra and characterize Toeplitz operators which belong to the algebra.

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1. Introduction

In the paper we study the C^*-algebra generated by Toeplitz operators T_a with piece-wise continuous symbols a acting on the Bergman space $\mathcal{A}^2(\mathbb{D})$ on the unit disk $\mathbb{D}$ in $\mathbb{C}$ (see Section 2 for exact definitions). Our aim here is to describe explicitly each operator from this algebra and to characterize all Toeplitz operators which belong to the algebra.

The first structural result on Toeplitz operator algebras is due to L. Coburn [5] and goes back to early 1970s. It says, that *the C^*-algebra $\mathcal{T}(C(\overline{\mathbb{D}}))$ generated by Toeplitz operators with symbols continuous on $\overline{\mathbb{D}}$ is irreducible and contains the entire ideal $\mathcal{K}$ of compact operators on $\mathcal{A}^2(\mathbb{D})$. Every operator $T \in \mathcal{T}(C(\overline{\mathbb{D}}))$ is of the form*

$$T = T_a + K\,,$$

where $a \in C(\overline{\mathbb{D}})$ and K is a compact operator.

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The key property of Toeplitz operators behind this result is that the semicommutator of two Toeplitz operators with continuous symbols $[T_a, T_b) = T_aT_b - T_{ab}$ is compact.

The maximal class of symbols for which the above structural result remains true was introduced and studied by K. Zhu [17]. This class is $Q = VMO_{\partial}(\mathbb{D}) \cap L_\infty(\mathbb{D})$ and is the maximal C^*-subalgebra of $L_\infty(\mathbb{D})$ having the compact semicommutator property.

However, for piece-wise continuous symbols, the semicommutators of Toeplitz operators are no longer compact in general. This immediately leads to a much more complicated structure fot the C^*-algebra generated by such operators. Indeed, apart from the initial generators, the Toeplitz operators T_a with piece-wise continuous symbols, the algebra contains now all elements of the form

$$\sum_{k=1}^{p} \prod_{j=1}^{q_k} T_{a_{j,k}} \,,$$

as well as the uniform limits of sequences of such elements.

We note that the description of the (Fredholm) symbol algebra for the C^*-algebra generated by Toeplitz operators T_a with piece-wise continuous symbols (see, for example, [13–15]) is well understood and known for a many years. At the same time many important questions connected with the structure of the Toeplitz operator algebra itself have remained unanswered since the very first work on this subject. We list some of them in the following general setting.

Let $\mathcal{A} \in L_\infty(\mathbb{D})$ be a set (linear space or algebra) of initial generating symbols. Denote by $\mathcal{T}(\mathcal{A})$ the C^*-algebra generated by all Toeplitz operators T_a with symbols from $\mathcal{A}$. The following questions are of great importance.

(i) Describe the (Fredholm) symbol algebra $\operatorname{Sym}\mathcal{T}(\mathcal{A}) = \mathcal{T}(\mathcal{A})/\mathcal{T}(\mathcal{A}) \cap \mathcal{K}$ of the algebra $\mathcal{T}(\mathcal{A})$, as well as the symbol homomorphism $\operatorname{sym} : \mathcal{T}(\mathcal{A}) \to \operatorname{Sym}\mathcal{T}(\mathcal{A})$; here $\mathcal{K}$ is the ideal of compact operators on $\mathcal{A}^2(\mathbb{D})$.
(ii) Describe a canonical representation of elements forming $\mathcal{T}(\mathcal{A})$, thus clarifying the structure of the algebra $\mathcal{T}(\mathcal{A})$.
(iii) Given an element $\operatorname{sym} A$ from the symbol algebra $\operatorname{Sym}\mathcal{T}(\mathcal{A})$, characterize an operator $A \in \mathcal{T}(\mathcal{A})$ having this (Fredholm) symbol.
(iv) Characterize the Toeplitz operators T_b which belong to $\mathcal{T}(\mathcal{A})$, as well as the variety of their possible symbols b.

In the paper we consider the case when $\mathcal{A}$ is a class of piece-wise continuous symbols (defined in Section 2). As we already mentioned, the complete answer to (i) is well known, while questions (ii)–(iv) remained unanswered. Our aim here is to answer to these last questions.

We mention that an intensive study has been recently devoted to the question of when the product of two Toeplitz operators is a Toeplitz operator. Not pretending to be complete, we cite, for example, the papers [1–4, 8, 9]. This interesting and important problem leads to a more general question: *under what* *conditions will applications all of algebraic operations* (*summation, product, uniform limit*) *to Toeplitz operators produce a Toeplitz operator; which combinations of which Toeplitz operators give Toeplitz operators?* If we restrict ourselves to a specific class of initial Toeplitz operators, then the last question becomes precisely the fourth one from the above list.

2. Symbol class and operators

Let $\mathbb{D}$ be the unit disk on the complex plane. Consider $L_2(\mathbb{D})$ with the standard Lebesgue plane measure $dv(z) = dxdy$, $z = x + iy \in \mathbb{D}$, and its Bergman subspace $\mathcal{A}^2(\mathbb{D})$ which consists of all functions analytic in $\mathbb{D}$. It is well known that the orthogonal Bergman projection B of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$ has the following form

$$(B\varphi)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)\, dv(\zeta)}{(1 - z\overline{\zeta})^2} \,.$$

Given a function $a \in L_\infty$, the Toeplitz operator T_a with symbol a is defined as follows

$$T_a : \ \varphi \in \mathcal{A}^2(\mathbb{D}) \longmapsto B(a\varphi) \in \mathcal{A}^2(\mathbb{D}) \,.$$

In the paper we study Toeplitz operators with piece-wise continuous symbols and the C^*-algebra generated by such operators. As was mentioned in [12], considering Toeplitz operators with piece-wise continuous symbols, it turns out that neither the curves supporting the symbol discontinuities nor the number of such curves meeting at a boundary point of discontinuity play any essential role for the Toeplitz operator algebra studied. We can start from very different sets of symbols and obtain exactly the same operator algebra as a result. Thus, without loss of generality, we fix now a certain setup which is suitable for our needs.

We fix a finite number of distinct points $T = \{t_1, \ldots, t_m\}$ on the boundary γ of the unit disk $\mathbb{D}$, and let

$$\delta = \min_{k \neq j}\{|t_k - t_j|,\, 1\} \,.$$

Denote by ℓ_k, $k = 1, \ldots, m$, the part of the radius of $\mathbb{D}$ starting at t_k and having length $\delta/3$; and let $\mathcal{L} = \bigcup_{k=1}^m \ell_k$. We denote by $PC(\overline{\mathbb{D}}, T)$ the set (algebra) of all piece-wise continuous functions on $\mathbb{D}$ which are continuous in $\overline{\mathbb{D}} \setminus \mathcal{L}$ and have one-sided limit values at each point of $\mathcal{L}$. In particular, every function $a \in PC(\overline{\mathbb{D}}, T)$ has at each point $t_k \in T$ two (different, in general) limit values:

$$a^-(t_k) = a(t_k - 0) = \lim_{\gamma \ni t \to t_k,\, t \prec t_k} a(t) \quad \text{and} \quad a^+(t_k) = a(t_k + 0) = \lim_{\gamma \ni t \to t_k,\, t \succ t_k} a(t) \,,$$

the signs $\pm$ here correspond to the standard orientation of the boundary γ of $\mathbb{D}$.

For each $k = 1, \ldots, m$, denote by $\chi_k = \chi_k(z)$ the characteristic function of the half-disk obtained by cutting $\mathbb{D}$ by the diameter passing through $t_k \in T$, and such that $\chi_k^+(t_k) = 1$, and thus $\chi_k^-(t_k) = 0$.

For each $k = 1, \dots, m$, we introduce two neighborhoods of the point t_k:

$$V_k' = \left\{ z \in \overline{\mathbb{D}} : |z - t_k| < \frac{\delta}{6} \right\} \quad \text{and} \quad V_k'' = \left\{ z \in \overline{\mathbb{D}} : |z - t_k| < \frac{\delta}{3} \right\},$$

and fix a continuous function $v_k = v_k(z) : \ \overline{\mathbb{D}} \to [0,1]$ such that

$$v_k|_{\overline{V_k'}} \equiv 1\,, \quad v_k|_{\overline{\mathbb{D}} \setminus V_k''} \equiv 0\,.$$

For easy reference we summarize three well known facts in the theory of Toeplitz operators in the next lemma.

Lemma 2.1. *The following properties hold:*

(i) *let $L_\infty^0(\mathbb{D})$ be the closure in $L_\infty(\mathbb{D})$ of the set of all L_∞-functions having compact support in $\mathbb{D}$; then for each function $a \in L_\infty^0(\mathbb{D})$ the Toeplitz operator T_a is compact;*

(ii) *for each pair of functions $a \in L_\infty(\mathbb{D})$ and $b \in C(\overline{\mathbb{D}})$ the semi-commutator $[T_a, T_b) = T_a T_b - T_{ab}$ is compact;*

(iii) *for each pair of functions $a \in L_\infty(\mathbb{D})$ and $b \in C(\overline{\mathbb{D}})$ the commutator $[T_a, T_b] = T_a T_b - T_b T_a$ is compact.*

We mention as well that for a, $b \in PC(\overline{\mathbb{D}}, T)$, the semi-commutator $[T_a, T_b)$ is not anymore compact, in general, while the commutator $[T_a, T_b]$ remains to be compact.

Using Lemma 2.1 it is easy to see that for any symbol $a \in PC(\overline{\mathbb{D}}, T)$, the Toeplitz operator T_a admits the canonical representations

$$\begin{aligned} T_a &= T_{s_a} + \sum_{k=1}^m T_{v_k} p_{a,k}(T_{\chi_k}) T_{v_k} + K \\ &= T_{s_a} + \sum_{k=1}^m T_{u_k} p_{a,k}(T_{\chi_k}) + K' \\ &= T_{s_a} + \sum_{k=1}^m p_{a,k}(T_{\chi_k}) T_{u_k} + K''\,, \end{aligned}$$

where $s_a(z)$ is a continuous function on $\overline{\mathbb{D}}$ such that the following restrictions on γ coincide:

$$s_a(z)|_\gamma \equiv \left[a(z) - \sum_{k=1}^m \Big[a^-(t_k) + \big(a^+(t_k) - a^-(t_k)\big) \chi_k(z) \Big] u_k(z) \right]_\gamma\,,$$

and where

$$p_{a,k}(x) = a^-(t_k) + \big(a^+(t_k) - a^-(t_k)\big) x = a^-(t_k)(1-x) + a^+(t_k) x\,, \quad k = 1, \dots, m\,,$$

are the first order polynomials in x, $u_k(z) = v_k(z)^2$, and K, K', K'' are compact operators.

Indeed, by the second statement of the lemma, each right hand side operator is a compact perturbation of the Toeplitz operator $T_{\widetilde{a}}$, where

$$\widetilde{a}(z) = s_a(z) + \sum_{k=1}^{m} \Big[a^-(t_k) + \big(a^+(t_k) - a^-(t_k)\big)\chi_k(z)\Big] u_k(z)\,.$$

We note that each function $\chi_k(z)u_k(z)$, $k = 1, \ldots, m$, belongs to $PC(\overline{\mathbb{D}}, T)$ and that $s_a(t_k) = 0$ for all $t_k \in T$. Then the difference $a(z) - \widetilde{a}(z)$ is continuous at every point of the boundary γ and $[a(z) - \widetilde{a}(z)]_\gamma \equiv 0$. Thus by the first statement of the lemma the difference $T_a - T_{\widetilde{a}}$ is compact.

Such representations are essentially unique in the sense that the values of $s_a(z)$ on γ are uniquely defined and if the function $s_a(z)$ is changed for another one with the same boundary values, the result will be altered at most by a compact operator.

3. Algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$

We denote by $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ the C^*-algebra generated by all Toeplitz operators T_a whose symbols a belong to $PC(\overline{\mathbb{D}}, T)$. It is well known that this algebra is irreducible and contains the entire ideal $\mathcal{K}$ of all compact on $\mathcal{A}^2(\mathbb{D})$ operators.

We give now the description (see, for details, [13–15]) of the (Fredholm) symbol algebra $\operatorname{Sym}\mathcal{T}(PC(\overline{\mathbb{D}}, T)) = \mathcal{T}(PC(\overline{\mathbb{D}}, T))/\mathcal{K}$ of the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$.

Let $\widehat{\gamma}$ be the boundary γ cut at the points $t_k \in T$. The pair of points of $\widehat{\gamma}$ which correspond to the point $t_k \in T$, $k = 1, \ldots, m$, will be denoted by $t_k - 0$ and $t_k + 0$, following the positive orientation of γ. Let $\overline{X} = \bigsqcup_{k=1}^m \Delta_k$ be the disjoint union of segments $\Delta_k = [0, 1]$. Denote by Γ the union $\widehat{\gamma} \cup \overline{X}$ with the following point identification

$$t_k - 0 \equiv 0_k\,, \quad t_k + 0 \equiv 1_k\,,$$

where $t_k \pm 0 \in \widehat{\gamma}$, 0_k and 1_k are the boundary points of Δ_k, $k = 1, \ldots, m$.

Theorem 3.1. *The symbol algebra* $\operatorname{Sym}\mathcal{T}(PC(\overline{\mathbb{D}}, T)) = \mathcal{T}(PC(\overline{\mathbb{D}}, T))/\mathcal{K}$ *of the algebra* $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ *is isomorphic and isometric to the algebra* $C(\Gamma)$. *The homomorphism*

$$\operatorname{sym}\,:\ \mathcal{T}\big(PC(\overline{\mathbb{D}}, T)\big) \longrightarrow \operatorname{Sym}\mathcal{T}\big(PC(\overline{\mathbb{D}}, T)\big) \cong C(\Gamma)$$

is generated by the mapping of generators of $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$

$$\operatorname{sym}\,: T_a \longmapsto \begin{cases} a(t), & t \in \widehat{\gamma} \\ a(t_k - 0)(1 - x) + a(t_k + 0)x\,, & x \in [0, 1] \end{cases},$$

where $t_k \in T$, $k = 1, 2, \ldots, m$.

The proof of the theorem is based on the standard local principle (see, for example, [6, 10, 11]), use the localization by the points of γ and the description of each local algebra $\widehat{\mathcal{T}}(t)$, $t \in \gamma$.

In what follows we will use two different descriptions of the local algebras $\widehat{\mathcal{T}}(t_k)$, for $t_k \in T \subset \gamma$, which we now proceed to describe.

As a Toeplitz operator T_a with symbol continuous at the point t_k is locally equivalent at the point t_k to the scalar operator $a(t_k)I = T_{a(t_k)}$, the local algebra $\widehat{\mathcal{T}}(t_k)$ is the C^*-algebra with identity generated by the single self-adjoint element T_{χ_k}, and thus is isomorphic and isometric to $C(\operatorname{sp} T_{\chi_k})$. It is well known that $\operatorname{sp} T_{\chi_k} = [0,1]$. Thus as the first description of the local algebra $\widehat{\mathcal{T}}(t_k)$ we have:

The local algebra $\widehat{\mathcal{T}}(t_k)$ is isomorphic and isometric to $C[0,1]$, and the isomorphism

$$\pi'_{t_k} : \widehat{\mathcal{T}}(t_k) \longrightarrow C[0,1]$$

is generated by the mapping $\pi'_{t_k} : T_{\chi_k} \mapsto x$, *where* $x \in \Delta_k = [0,1]$.

For the second description we construct a unitary operator directly reducing T_{χ_k} to a multiplication operator.

Let Π be the upper half-plane in $\mathbb{C}$. Introduce the Möbius transformation

$$\alpha_k(z) = i\frac{t_k - z}{z + t_k}\,,$$

which maps the unit disk $\mathbb{D}$ onto Π, sending the point t_k to 0 and the opposite point $-t_k$ to ∞. We introduce the space $L_2(\Pi)$, with the usual Lebesgue plane measure, and its Bergman subspace $\mathcal{A}^2(\Pi)$ which consists of all functions analytic in Π. Then

$$(V_k\varphi)(z) = -\frac{2it_k}{(z+t_k)^2}\,\varphi\left(i\frac{t_k - z}{z + t_k}\right) \tag{3.1}$$

is obviously a unitary operator both from $L_2(\Pi)$ onto $L_2(\mathbb{D})$, and from $\mathcal{A}^2(\Pi)$ onto $\mathcal{A}^2(\mathbb{D})$, and its inverse (and adjoint) has the form

$$(V_k^{-1}\varphi)(w) = -\frac{2it_k}{(w+i)^2}\,\varphi\left(t_k\frac{i-w}{w+i}\right).$$

It is a simple calculation to check that

$$V_k T_{\chi_k} V_k^{-1} = T_{\chi_+}\,,$$

where χ_+ is the characteristic function of the right quarter-plane in Π.

We denote by $L_\infty^{\{0,\pi\}}(0,\pi)$ the C^*-subalgebra of $L_\infty(0,\pi)$ of all functions having limits at the points 0 and π. And let $H(L_\infty^{\{0,\pi\}}(0,\pi))$ be the algebra which consists of all homogeneous functions of zero order on the upper half-plane whose restrictions onto the upper half of the unit circle (parameterized by $\theta \in [0,\pi]$) belong to $L_\infty^{\{0,\pi\}}(0,\pi)$. Further let $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ be the C^*-algebra generated by all Toeplitz operators T_a with symbols $a \in H(L_\infty^{\{0,\pi\}}(0,\pi))$.

The function χ_+ obviously belongs to $H(L_\infty^{\{0,\pi\}}(0,\pi))$ and thus $T_{\chi_+} \in \mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$. Moreover, as shown in [12], the Toeplitz operator T_{χ_+} generates the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$.

The exact result is as follows.

Passing to polar coordinates on the upper half-plane Π we have

$$L_2(\Pi) = L_2(\mathbb{R}_+, rdr) \otimes L_2\big([0,\pi], d\theta\big) = L_2(\mathbb{R}_+, rdr) \otimes L_2(0,\pi)\,.$$

We introduce (see [16]) two operators: the unitary operator

$$U = M \otimes I : L_2(\mathbb{R}_+, rdr) \otimes L_2(0,\pi) \longrightarrow L_2(\mathbb{R}) \otimes L_2(0,\pi)\,,$$

where the Mellin transform $M : L_2(\mathbb{R}_+, rdr) \longrightarrow L_2(\mathbb{R})$ is given by

$$(M\psi)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} r^{-i\lambda}\, \psi(r)\, dr\,,$$

and the isometric imbedding $R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}_1^2 \subset L_2(\mathbb{R} \times [0,\pi])$, which is given by

$$(R_0 f)(\lambda,\theta) = f(\lambda) \cdot \sqrt{\frac{2\lambda}{1-e^{-2\pi\lambda}}}\; e^{-(\lambda+i)\theta}\,.$$

The adjoint operator $R_0^* : L_2(\mathbb{R} \times [0,\pi]) \longrightarrow L_2(\mathbb{R})$ has the form

$$(R_0^*\psi)(\lambda) = \sqrt{\frac{2\lambda}{1-e^{-2\pi\lambda}}} \int_0^\pi \psi(\lambda,\theta)\, e^{-(\lambda-i)\theta}\, d\theta\,.$$

Now the operator $R = R_0^* U$ maps the space $L_2(\Pi)$ onto $L_2(\mathbb{R})$, and its restriction

$$R|_{\mathcal{A}^2(\Pi)} : \mathcal{A}^2(\Pi) \longrightarrow L_2(\mathbb{R})$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2(\Pi) \subset L_2(\Pi)$$

is an isometric isomorphism of $L_2(\mathbb{R})$ onto the Bergman subspace $\mathcal{A}^2(\Pi)$ of the space $L_2(\Pi)$.

Theorem 3.2 ([16]). *Let $a = a(\theta) \in H(L_\infty^{\{0,\pi\}}(0,\pi))$. Then the Toeplitz operator T_a, acting on $\mathcal{A}^2(\Pi)$, is unitary equivalent to the multiplication operator $\gamma_a I = R T_a R^*$, acting on $L_2(\mathbb{R})$. The function $\gamma_a(\lambda)$ is given by*

$$\gamma_a(\lambda) = \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^\pi a(\theta)\, e^{-2\lambda\theta}\, d\theta\,, \quad \lambda \in \overline{\mathbb{R}}\,, \tag{3.2}$$

and belongs to $C(\overline{\mathbb{R}})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is the two-point compactification of $\mathbb{R}$.

In particular, for $a = \chi_+$, we have (see [12])

$$\gamma_{\chi_+}(\lambda) = \frac{1}{e^{-\pi\lambda}+1}\,, \quad \lambda \in \overline{\mathbb{R}}\,,$$

and

$$T_{\chi_+} = R^* \gamma_{\chi_+}(\lambda) R\,.$$

Theorem 3.3 ([12]). *The C^*-algebra with identity generated by T_{χ_+} coinsides with the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ and is isomorphic and isometric to $C(\overline{\mathbb{R}})$. The isomorphism*

$$\pi_+ : \mathcal{T}\Big(H\big(L_\infty^{\{0,\pi\}}(0,\pi)\big)\Big) \longrightarrow C(\overline{\mathbb{R}})$$

is generated by the mapping $\pi_+ :\ T_{\chi_+} \longmapsto \gamma_{\chi_+}(\lambda)$.

To obtain the second description of the local algebra $\widehat{\mathcal{T}}(t_k)$ consider the unitary operator $U_k = RV_k$, and note that

$$U_k T_{\chi_k} U_k^{-1} = \gamma_{\chi_+}(\lambda) \quad \text{or} \quad T_{\chi_k} = U_k^{-1}\gamma_{\chi_+}(\lambda)U_k \,. \tag{3.3}$$

Thus we have: *The local algebra $\widehat{\mathcal{T}}(t_k)$ is isomorphic and isometric to $C(\overline{\mathbb{R}})$ and the isomorphism*

$$\pi''_{t_k} : \ \widehat{\mathcal{T}}(t_k) \longrightarrow C(\overline{\mathbb{R}})$$

is generated by the mapping $\pi''_{t_k} : T_{\chi_k} \mapsto \gamma_{\chi_+}(\lambda)$, *where* $\lambda \in \overline{\mathbb{R}}$.

We summarize the above in the next proposition.

Proposition 3.4. *For each point $t_k \in T$, the local algebra $\widehat{\mathcal{T}}(t_k)$ consists of all operators of the form $f(T_{\chi_k})$, where $f \in C[0,1]$. Each such operator admits the representation*

$$f(T_{\chi_k}) = U_k^{-1} f\big(\gamma_{\chi_+}(\lambda)\big)U_k \,.$$

4. Operators of the algebra $\mathcal{T}(PC(\overline{\mathbb{D}},T))$

As has been already mentioned, the algebra $\mathcal{T}(PC(\overline{\mathbb{D}},T))$, apart from its initial generators T_a with $a \in PC(\overline{\mathbb{D}},T)$, contains all elements of the form

$$\sum_{k=1}^{p}\prod_{j=1}^{q_k} T_{a_{j,k}} \,, \tag{4.1}$$

as well as the uniform limits of sequences of such elements. Our aim here is to characterize each operator from the algebra $\mathcal{T}(PC(\overline{\mathbb{D}},T))$ up to a compact summand.

We start with the following lemma.

Lemma 4.1. *For each $n \in \mathbb{N}$ and each $k = 1,\ldots,m$, there is a function $s_{n,k} = s_{n,k}(z) \in C(\overline{\mathbb{D}})$ and a compact operator $K_{n,k}$ such that*

$$(T_{v_k}T_{\chi_k}T_{v_k})^n = T_{v_k}T^n_{\chi_k}T_{v_k} + T_{s_{n,k}} + K_{n,k} \,.$$

Proof. We have obviously

$$\begin{aligned}(T_{v_k}T_{\chi_k}T_{v_k})^n &= T_{v_k^n}T^n_{\chi_k}T_{v_k^n} + K' \,,\\ T_{v_k}T^n_{\chi_k}T_{v_k} &= \big(T_{v_k^{1/n}}T_{\chi_k}T_{v_k^{1/n}}\big)^n + K'' \,,\end{aligned}$$

where K' and K'' are compact operators. Thus both operators $T_{v_k^n}T^n_{\chi_k}T_{v_k^n}$ and $T_{v_k}T^n_{\chi_k}T_{v_k}$ belong to the algebra $\mathcal{T}(PC(\overline{\mathbb{D}},T))$. Calculating their symbols we have that

$$\text{sym}\,\big(T_{v_k^n}T^n_{\chi_k}T_{v_k^n} - T_{v_k}T^n_{\chi_k}T_{v_k}\big) = s_{n,k}(t) \,,$$

where the continuous function $s_{n,k}(t)$ on γ has the form

$$s_{n,k}(t) = \begin{cases} \big[v_k^{2n}(t) - v_k^2(t)\big]\chi_k(t) \,, & t \in \widehat{\gamma} \setminus \big(V'_k \cup (\overline{\mathbb{D}} \setminus V''_k)\big) \,,\\ 0 \,, & t \in \widehat{\gamma} \cap \big(V'_k \cup (\overline{\mathbb{D}} \setminus V''_k)\big) \,.\end{cases}$$

which is a continuous function on γ. Extending $s_{n,k}(t)$ to a continuous function on $\overline{\mathbb{D}}$ and returning from symbols to operators we obtain the desired property. □

Corollary 4.2. *For every polynomial* $p(x)$ *and each* $k = 1, \dots, m$, *the operator* $A_{p,k} = T_{v_k} p(T_{\chi_k}) T_{v_k}$ *belongs to the algebra* $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$, *and*

$$(\operatorname{sym} A_{p,k})|_{\Delta_k} = p(x)\,, \quad x \in [0,1]\,.$$

Corollary 4.3. *Each operator* A *of the form* (4.1) *admits the canonical representation*

$$A = \sum_{i=1}^{p} \prod_{j=1}^{q_i} T_{a_{i,j}} = T_{s_A} + \sum_{k=1}^{m} T_{v_k} p_{A,k}(T_{\chi_k}) T_{v_k} + K_A\,,$$

where $s_A = s_A(z) \in C(\overline{\mathbb{D}})$, $p_{A,k} = p_{A,k}(x)$, $k = 1, \dots, m$, *are polynomials, and* K_A *is a compact operator.*

Lemma 4.4. *Let* $f \in C[0,1]$. *Then for each* $k = 1, \dots, m$ *the operator* $A_{f,k} = T_{v_k} f(T_{\chi_k}) T_{v_k}$ *belongs to the algebra* $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$, *and*

$$(\operatorname{sym} A_{f,k})|_{\Delta_k} = f(x)\,, \quad x \in [0,1]\,.$$

Proof. Recall that the operator T_{χ_k} is self-adjoint and its spectrum is equal to $[0,1]$. Let $\{p_n(x)\}_{n\in\mathbb{N}}$ be a sequence of polynomials which converges uniformly on $[0,1]$ to the function $f(x)$. Then by the standard functional calculus in a C^*-algebra we have

$$\|p_n(T_{\chi_k}) - f(T_{\chi_k})\| = \sup_{x\in[0,1]} |p_n(x) - f(x)|\,,$$

and thus the operator $A_{f,k}$ is the uniform limit of the sequence $\{T_{v_k} p_n(T_{\chi_k}) T_{v_k}\}_{n\in\mathbb{N}}$ of elements of the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$.

Finally, the restriction $(\operatorname{sym} A_{f,k})|_{\Delta_k}$ coincides with the uniform limit of the restrictions $(\operatorname{sym} T_{v_k} p_n(T_{\chi_k}) T_{v_k})|_{\Delta_k} = p_n(x)$, $x \in [0,1]$, thus giving the desired result. □

The next theorem starts the characterization of operators from the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ representing them in certain canonical forms.

Theorem 4.5. *Every operator* $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ *admits the canonical representations*

$$\begin{aligned} A &= T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K \\ &= T_{s_A} + \sum_{k=1}^{m} T_{u_k} f_{A,k}(T_{\chi_k}) + K' \\ &= T_{s_A} + \sum_{k=1}^{m} f_{A,k}(T_{\chi_k}) T_{u_k} + K''\,, \end{aligned}$$

where $s_A(z)$ *is a continuous function on* $\overline{\mathbb{D}}$, $u_k(z) = v_k(z)^2$, $f_{A,k}(x)$, $k = 1, \dots, m$, *are continuous functions on* $[0,1]$, *and* K, K', K'' *are compact operators.*

Before we pass to the proof, we note that such representations have already been obtained in Section 2 for the generators T_a, where $a \in PC(\overline{\mathbb{D}}, T)$, of the algebra $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$.

As in Section 2, these representations are essentially unique in the sense that the values of $s_A(z)$ on γ and the functions $f_{A,k}(x)$ are uniquely defined by the operator A, and if the function $s_A(z)$ is changed for another one with the same boundary values, the result will be altered at most by a compact operator.

Proof. We will show the first representation only; the other two follow from the fact that operators from the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ commute modulo a compact operator.

Hence, given an operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$, we introduce the functions $f_{A,k}(x) \in C[0,1]$, $k = 1, \ldots, m$, by

$$f_{A,k}(x) = (\operatorname{sym} A)|_{\Delta_k}\,, \quad x \in [0,1]\,.$$

Then the symbol of the operator $A - \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k}$ has the form

$$\operatorname{sym}\left(A - \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k}\right) = \begin{cases} s_A(t) & t \in \widehat{\gamma} \\ 0\,, & x \in \Delta_k\,,\ k = 1, \ldots m \end{cases},$$

where

$$s_A(t) = (\operatorname{sym} A)(t) - \sum_{k=1}^{m} v_k^2(t) \Big[f_{A,k}(0)\big(1 - \chi_k(t)\big) + f_{A,k}(1)\chi_k(t) \Big]$$

is a continuous function on γ, and such that $s_A(t_k) = 0$ for all $t_k \in T$.

To finish the proof we extend s_A to a continuous function on $\overline{\mathbb{D}}$ and return from symbols to operators. □

Theorem 4.5 and Proposition 3.4 lead to the next characterization of elements of the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$.

Corollary 4.6. *Every operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ admits the representations*

$$\begin{aligned} A &= T_{s_A} + \sum_{k=1}^{m} T_{v_k} U_k^{-1} f_{A,k}\big(\gamma_{\chi_+}(\lambda)\big) U_k T_{v_k} + K \\ &= T_{s_A} + \sum_{k=1}^{m} T_{u_k} U_k^{-1} f_{A,k}\big(\gamma_{\chi_+}(\lambda)\big) U_k + K' \\ &= T_{s_A} + \sum_{k=1}^{m} U_k^{-1} f_{A,k}\big(\gamma_{\chi_+}(\lambda)\big) U_k T_{u_k} + K''\,, \end{aligned}$$

where $s_A(z)$ is a continuous function on $\overline{\mathbb{D}}$ whose restriction to γ is given by

$$s_A(t) = (\operatorname{sym} A)(t) - \sum_{k=1}^{m} u_k(t) \Big[f_{A,k}(0)\big(1 - \chi_k(t)\big) + f_{A,k}(1)\chi_k(t) \Big]\,,$$

where $f_{A,k}(x) = (\operatorname{sym} A)|_{\Delta_k}$, *the operators* U_k *are defined in* (3.3), $u_k(x) = v_k^2(x)$, $k = 1, \ldots, m$, *and* K, K', K'' *are compact operators.*

5. Toeplitz operators of the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$

In this section we show that, apart from the initial generators, the C^*-algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$ contains many other (non compact) Toeplitz operators which are drastically different from the initial generators. By Toeplitz operator here we always mean a Toeplitz operator with *bounded measurable* symbol.

Let A be an operator of the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T))$. By Theorem 4.5 it admits the canonical representation

$$A = T_{s_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K \, .$$

Lemma 5.1. *The operator* A *is a compact perturbation of a Toeplitz operator if and only if each operator* $T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k}$, $k = 1, \ldots, m$, *is a compact perturbation of a Toeplitz operator.*

Proof. The "if" part is obvious. To prove the "only if" part, we assume that $A = T_a + K_1$ for some $a \in L_\infty(\mathbb{D})$. Using v_k in place of $v_k^{1/2}$ in Theorem 4.5 we represent the operator A in its second canonical form

$$A = T_{s'_A} + \sum_{k=1}^{m} T_{v_k} f_{A,k}(T_{\chi_k}) + K' \, .$$

Then, multiplying by T_{v_k} and using statement (ii) of Lemma 2.1, we have

$$A T_{v_k} = T_{s'_A v_k} + T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} + K_2 = T_{a v_k} + K_3 \, ,$$

or

$$T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} = T_{a v_k - s'_A v_k} + (K_3 - K_2) \, . \qquad \square$$

The result of Lemma 5.1 obviously remains true if we change the operators $T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k}$ for either $T_{u_k} f_{A,k}(T_{\chi_k})$, or $f_{A,k}(T_{\chi_k}) T_{u_k}$, $k = 1, \ldots, m$.

Theorem 5.2. *For any* $k = 1, \ldots, m$, *the operator* $T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k}$ *is a compact perturbation of a Toeplitz operator if and only if the operator* $f_{A,k}(T_{\chi_k})$ *is a compact perturbation of a Toeplitz operator.*

Proof. The "if" part is again obvious. To prove the "only if" part, we first reduce the problem to the real valued function $f_{A,k}$. To this end we assume that

$$T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k} = T_a + K_1 \, , \quad \text{for some} \quad a \in L_\infty(\mathbb{D}) \, .$$

Passing to adjoint operators and taking into account that the functions v_k and χ_k are real valued, we have

$$\left(T_{v_k} f_{A,k}(T_{\chi_k}) T_{v_k}\right)^* = T_{v_k} \overline{f}_{A,k}(T_{\chi_k}) T_{v_k} = T_{\overline{a}} + K_2 \, .$$

Summing up these equalities we have that $T_{v_k}(\mathrm{Re}f_{A,k})(T_{\chi_k})T_{v_k}$ is a compact perturbation of a Toeplitz operator. Subtracting the equalities, we have that $T_{v_k}(\mathrm{Im}f_{A,k})(T_{\chi_k})T_{v_k}$ is a compact perturbation of a Toeplitz operator as well.

That is, the operator $T_{v_k}f_{A,k}(T_{\chi_k})T_{v_k}$ is a compact perturbation of a Toeplitz operator if and only if both $T_{v_k}(\mathrm{Re}f_{A,k})(T_{\chi_k})T_{v_k}$ and $T_{v_k}(\mathrm{Im}f_{A,k})(T_{\chi_k})T_{v_k}$ are compact perturbations of Toeplitz operators. Thus proving the part "only if" we can assume that the function $f_{A,k}$ is real valued, moreover we can consider the operator $f_{A,k}(T_{\chi_k})T_{u_k}$ instead of $T_{v_k}f_{A,k}(T_{\chi_k})T_{v_k}$.

We note as well that without loss of generality we may assume in what follows that $t_k = i \in \gamma$, because otherwise, using an appropriate rotation, we come to the unitary equivalent operator with $t_k = i \in \gamma$. Hence, let $t_k = i$ and let $f_{A,k}$ be a real valued function such that $f_{A,k}(T_{\chi_k})T_{u_k} = T_a + K_1$ for some $a \in L_\infty(\mathbb{D})$.

We introduce now the operator

$$(Z\varphi)(z) = \overline{\varphi(\overline{z})}\,,$$

which is obviously unitary on both $L_2(\mathbb{D})$ and $\mathcal{A}^2(\mathbb{D})$. Then, as is easy to see,

$$Zf_{A,k}(T_{\chi_k(z)})T_{u_k(z)}Z = f_{A,k}(T_{\chi_k(z)})T_{u_k(\overline{z})} = T_{\overline{a(\overline{z})}} + K_2\,,$$

and thus

$$f_{A,k}(T_{\chi_k(z)})\left(T_{u_k(z)} + T_{u_k(\overline{z})}\right) = T_b + K_3\,,$$

where $b(z) = a(z) + \overline{a(\overline{z})}$, or

$$f_{A,k}(T_{\chi_k(z)}) = f_{A,k}(T_{\chi_k(z)})\left(I - T_{u_k(z)} - T_{u_k(\overline{z})}\right) + T_b + K_3\,.$$

The operator $f_{A,k}(T_{\chi_k(z)})\left(I - T_{u_k(z)} - T_{u_k(\overline{z})}\right)$ obviously belongs to the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, T'))$ with $T' = \{i, -i\}$ (that is, we have only two points of symbol discontinuity: $t_1 = i$ and $t_2 = -i$) and its symbol is a continuous function on γ (that is, a continuous function on Γ which is constant on each Δ_j, $j = 1, 2$) and identically equals to 0 at $\gamma \cap (V'_k \cup \overline{V'_k}) = \gamma \cap (V'_1 \cup V'_2)$. Thus the operator $f_{A,k}(T_{\chi_k(z)})\left(I - T_{u_k(z)} - T_{u_k(\overline{z})}\right)$ is a compact perturbation of some Toeplitz operator T_c with continuous symbol c, and thus we have finally

$$f_{A,k}(T_{\chi_k}) = T_{b+c} + K_3\,. \qquad \square$$

By Proposition 3.4 every operator of the form $f(T_{\chi_k})$, with $f \in C[0,1]$, is unitary equivalent to the multiplication operator $(f \circ \gamma_+)I$. That is, the C^*-algebra generated by (and consisting of) all such operators intersects the ideal $\mathcal{K}$ of compact operators in just the zero operator. This implies that an operator of the form $f(T_{\chi_k})$ is a compact perturbation of a Toeplitz operator if and only if it is a Toeplitz operator itself.

Summarizing the above we come to the main result of the section.

Theorem 5.3. *An operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ is a compact perturbation of a Toeplitz operator if and only if every operator $f_{A,k}(T_{\chi_k})$ is a Toeplitz operator, where $f_{A,k} = (\mathrm{sym}\,A)|_{\Delta_k}$ and $k = 1, \ldots, m$.*

The next theorem gives the description of the symbol of a Toeplitz operator for the case when $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ is of the form $A = T_a + K$.

Theorem 5.4. *Let $A = T_a + K$. Thus all the operators $(\operatorname{sym} A)|_{\Delta_k}(T_{\chi_k})$, where $k = 1, \dots, m$, are Toeplitz, i.e., $(\operatorname{sym} A)|_{\Delta_k}(T_{\chi_k}) = T_{a_k}$ for some $a_k \in L_\infty(\mathbb{D})$. Then the symbol a of the operator T_a is given by*

$$a(z) = s_A(z) + \sum_{k=1}^{m} a_k(z) v_k^2(z)\,,$$

where $s_A(z)$ is a continuous function on $\overline{\mathbb{D}}$ whose restriction to γ coincides with

$$s_A(t) = (\operatorname{sym} A)(t) - \sum_{k=1}^{m} \Big[(\operatorname{sym} A)|_{\Delta_k}(0)\big(1 - \chi_k(t)\big) + (\operatorname{sym} A)|_{\Delta_k}(1)\chi_k(t)\Big] v_k^2(t)\,.$$

Proof. Follows directly from Corollary 4.6. □

Note that the operators $f(T_{\chi_k})$ and $f(T_{\chi_+})$, being unitary equivalent, can be Toeplitz operators only simultaneously. That is, the question *whether an operator $A \in \mathcal{T}(PC(\overline{\mathbb{D}}, T))$ is a compact perturbation of a Toeplitz operator* reduces to the description of the Toeplitz operators in the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$. By Theorem 3.3 this algebra can be generated by T_+ alone, and thus consists of all operators of the form $f(T_{\chi_+})$, where $f \in C[0,1]$.

The known result on Toeplitz operators in the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ is contained in the next proposition.

Proposition 5.5 ([12]). *For any symbol $a = a(\theta) \in H(L_\infty^{\{0,\pi\}}(0,\pi))$, the Toeplitz operator T_a belongs to the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$, and is the following function of the operator T_{χ_+},*

$$T_a = f_a(T_{\chi_+})\,,$$

where

$$f_a(x) = \frac{2x^2}{\pi}\,\frac{\ln(1-x) - \ln x}{(1-x) - x} \int_0^\pi a(\theta) \left(\frac{1-x}{x}\right)^{\frac{2\theta}{\pi}} d\theta\,.$$

For a number of specific examples of symbols $a = a(\theta) \in H(L_\infty^{\{0,\pi\}}(0,\pi))$ and corresponding functions $f_a \in C[0,1]$, see [12].

Corollary 5.6. *For each function $a = a(w) = a(e^{i\theta}) \in H(L_\infty^{\{0,\pi\}}(0,\pi))$, where $w = re^{i\theta} \in \Pi$, and each $k = 1, \dots, m$, the Toeplitz operator*

$$T_{b_k} = V_k T_a V_k^{-1}$$

belongs to the algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, \{t_k, -t_k\})$ and has the symbol

$$b_k(z) = a\big(\alpha_k(z)\big) = a\left(i\frac{t_k - z}{z + t_k}\right).$$

Here the operator V_k is given by (3.1).

We can describe now the symbols of a wide variety of Toeplitz operators in $\mathcal{T}(PC(\overline{\mathbb{D}},T))$ which are drastically different from the initial generators. All of them have at each point of discontinuity $t_k \in T$, in general, infinitely many limit values reached by the hypercycles starting at t_k (i.e. the images under the Möbius transformation α_k^{-1} of rays on the upper half-plane Π starting at origin) and parameterized by functions from $L_\infty^{\{0,\pi\}}(0,\pi)$. We note that each of these (bounded) symbols b have one-sided limit values at the point t_k and these limit values coincide with the values of $\operatorname{sym} T_b$ at the endpoints of Δ_k:

$$b(t_k - 0) = (\operatorname{sym} T_b)(0_k)\,, \quad b(t_k + 0) = (\operatorname{sym} T_b)(1_k)\,.$$

Corollary 5.7. *For every function $a_k = a_k(w) = a_k(e^{i\theta}) \in H(L_\infty^{\{0,\pi\}}(0,\pi))$, where $w = re^{i\theta} \in \Pi$, $k = 1,\ldots,m$, and every function $s(z) \in L_\infty(\mathbb{D})$ having limits at all points of γ and such that $s|_\gamma \in C(\gamma)$, the Toeplitz operator T_b with symbol*

$$b(z) = s(z) + \sum_{k=1}^{m} a_k \left(i\frac{t_k - z}{z + t_k} \right) u_k(x)$$

belongs to the algebra $\mathcal{T}(PC(\overline{\mathbb{D}},T))$.

6. More Toeplitz operators

In the previous section we reduced the description of Toeplitz operators in the algebra $\mathcal{T}(PC(\overline{\mathbb{D}},T))$ to the description of Toeplitz operators in $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$. We show now that the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ contains many more Toeplitz operators than described by Proposition 5.5. Indeed, as we will see, it also contains (bounded) Toeplitz operators whose generally unbounded symbols $a(\theta)$ may not have limits at the endpoints 0 and π of the segment $[0,\pi]$.

We recall that the Toeplitz operator T_a with symbol $a(\theta)$ belongs to the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ if and only if the function $\gamma_a(\lambda)$, defined by (3.2), belongs to $C(\overline{\mathbb{R}})$.

Remark 6.1. Given a symbol $a(\theta)$, in what follows we will study the behavior of the corresponding function $\gamma_a(\lambda)$ when $\lambda \to \pm\infty$. It is clear that the behavior of $a(\theta)$ near the point 0, or π, determines the behavior of $\gamma_a(\lambda)$ near the point $+\infty$, or $-\infty$, respectively. The equality

$$\begin{aligned}\gamma_{a(\theta)}(-\lambda) &= \frac{-2\lambda}{1-e^{2\pi\lambda}} \int_0^\pi a(\theta)e^{2\lambda\theta}d\theta = \frac{-2\lambda}{1-e^{2\pi\lambda}} \int_0^\pi a(\pi-\theta)e^{2\lambda(\pi-\theta)}d\theta \\ &= \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^\pi a(\pi-\theta)e^{-2\lambda\theta}d\theta = \gamma_{a(\pi-\theta)}(\lambda)\end{aligned}$$

permits us to reduce this study to only one case, say considering the symbol $a(\theta)$ in a neighborhood of 0 and $\gamma_a(\lambda)$ in a neighborhood of $+\infty$.

We continue to consider the homogeneous symbols of zero order on the upper half-plane Π identifying them with functions $a(\theta)$, where $\theta \in [0,\pi]$.

For any L_1-symbol $a(\theta)$ we define the following averaging functions, which correspond to the endpoints of $[0,\pi]$,

$$C_a^{(1)}(\theta) = \int_0^{\theta} a(u)du\,, \quad D_a^{(1)}(\theta) = \int_{\pi-\theta}^{\pi} a(u)du$$

and

$$C_a^{(p)}(\theta) = \int_0^{\theta} C_a^{(p-1)}(u)du\,, \quad D_a^{(p)}(\theta) = \int_{\pi-\theta}^{\pi} D_a^{(p-1)}(u)du\,,$$

for each $p = 2, 3, \ldots$.

The next theorem gives the conditions on the behavior of L_1-symbols near endpoints 0 and π guaranteeing that the corresponding Toeplitz operators belong to the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$.

Theorem 6.2. *Let $a(\theta) \in L_1(0,\pi)$ and suppose that for some $p, q \in \mathbb{N}$,*

$$\lim_{\theta\to 0} \theta^{-p} C_a^{(p)}(\theta) = c_p \ (\in \mathbb{C}) \quad \text{and} \quad \lim_{\theta\to\pi} \theta^{-q}\, D_a^{(q)}(\theta) = d_q \ (\in \mathbb{C})\,. \tag{6.1}$$

Then $\gamma_a(\lambda) \in C(\overline{\mathbb{R}})$.

Proof. Consider first the case when $p = 1$ and $\lambda \to +\infty$. Integrating by parts we have

$$\begin{aligned}\gamma_a(\lambda) &= \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^{\pi} e^{-2\lambda\theta}\, dC_a^{(1)}(\theta) \\ &= \frac{2\lambda\, e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}}\, C_a^{(1)}(\pi) + \frac{4\lambda^2}{1-e^{-2\pi\lambda}} \int_0^{\pi} C_a^{(1)}(\theta)\, e^{-2\lambda\theta}\, d\theta\,.\end{aligned}$$

Taking into account the first equality in (6.1), we have

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$$\begin{aligned}\gamma_a(\lambda) &= \frac{2\lambda\, e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}}\, C_a^{(1)}(\pi) \\ &\quad + \frac{4\lambda^2\, c_1}{1-e^{-2\pi\lambda}} \int_0^{\pi} \theta\, e^{-2\lambda\theta}\, d\theta + \frac{4\lambda^2\, c_1}{1-e^{-2\pi\lambda}} \int_0^{\pi} \alpha(\theta)\theta\, e^{-2\lambda\theta}\, d\theta \\ &= := I_1(\lambda) + I_2(\lambda) + I_3(\lambda)\,,\end{aligned}$$

where $\lim_{\theta\to 0} \alpha(\theta) = 0$.

It is obvious that for sufficiently large λ, $|I_1(\lambda)| < \varepsilon$. Then,

$$I_2(\lambda) = c_1 - \frac{2\pi\lambda\, e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}}\,,$$

and thus for sufficiently large λ, $|I_2(\lambda) - c_1| < \varepsilon$.

To estimate I_3, we select a sufficiently small δ to guarantee that

$$\sup_{\theta\in(0,\delta)} |\alpha(\theta)| < \varepsilon\,.$$

Then

$$\begin{aligned}
|I_3(\lambda)| &\leq \text{const}\, \lambda^2 \left(\int_0^\delta \alpha(\theta)\theta\, e^{-2\lambda\theta}\, d\theta + \int_\delta^\pi \theta\, e^{-2\lambda\theta}\, d\theta \right) \\
&\leq \text{const} \left(\lambda^2 \varepsilon \int_0^\delta \theta\, e^{-2\lambda\theta}\, d\theta + \lambda^2 e^{-2\lambda\delta} \int_\delta^\pi \theta\, d\theta \right) \\
&\leq \text{const} \left(\varepsilon + \lambda^2 e^{-2\lambda\delta} \right).
\end{aligned}$$

That is, for sufficiently large λ we have as well that $|I_3(\lambda)| < \text{const}\,\varepsilon$, and the above three inequalities yield

$$\lim_{\lambda\to+\infty} \gamma_a(\lambda) = c_1 \,.$$

The case when $q = 1$ and $\lambda \to -\infty$ follows from Remark 6.1 and the case just considered. The continuity of $\gamma_a(\lambda)$ in all interior points of $[0, \pi]$ is obvious.

The proof for the cases when $p > 1$ and $q > 1$ is quite analogous and requires repeated (p-times, or q-times) integration by parts. □

We give now several examples of symbols bounded or unbounded near the endpoints of $[0, 1]$ and which oscillate approaching the endpoints.

Example. Let

$$a(\theta) = \theta^{-\beta} \sin\theta^{-\alpha}\,, \quad \text{where} \quad 0 \leq \beta < 1\,, \quad \alpha > 0\,. \tag{6.2}$$

This symbol oscillates near 0, is bounded when $\beta = 0$, is unbounded for all $\beta \in (0, 1)$, and is continuous at the another endpoint π for all admissible values of the parameters. That is we need to analyze the behavior of $a(\theta)$ near the point 0 only.

According to calculations of Example 4.4 in [7] we have that

$$C_a^{(1)}(\theta) = \frac{\theta^{\alpha-\beta+1}}{\alpha} \cos\theta^{-\alpha} + O(\theta^{2\alpha-\beta+1})\,, \quad \text{when} \quad \theta \to 0\,. \tag{6.3}$$

Thus, if $\alpha > \beta$ then

$$\lim_{\theta\to 0} \theta^{-1}\, C_a^{(1)}(\theta) = 0\,,$$

and the first condition in (6.1) is satisfied for $p = 1$.

Further, if $\alpha \leq \beta$ we need to consider the averages of the higher order. Indeed, formula (6.3) implies that

$$C_a^{(2)}(\theta) = O(\theta^{2\alpha-\beta+2})\,, \quad \text{when} \quad \theta \to 0$$

and, more generally, that

$$C_a^{(p)}(\theta) = O(\theta^{p\alpha-\beta+p})\,, \quad \text{when} \quad \theta \to 0\,.$$

Thus for each $\alpha \leq \beta$ there is $p_0 \in \mathbb{N}$ such that $p_0\alpha > \beta$, and thus the first condition in (6.1) is satisfied for $p = p_0$.

That is, the Toeplitz operator T_a with symbol (6.2) does belong to the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0, \pi)))$ for all admissible values of the parameters.

Example. Let

$$a(\theta) = (\sin\theta)^{-\beta} \sin(\sin\theta)^{-\alpha}, \quad \text{where} \quad 0 \le \beta < 1, \quad \alpha > 0. \tag{6.4}$$

This symbol oscillates near both endpoints of $[0,1]$, is bounded when $\beta = 0$, and is unbounded for all $\beta \in (0,1)$.

Analogously to the previous example one can show that if $p_0\alpha > \beta$ then both conditions in (6.1) are satisfied for $p = p_0$, and thus the Toeplitz operator T_a with symbol (6.4) belongs to the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ as well.

We show now that not all oscillating symbols, even bounded and continuous, generate the Toeplitz operators which belong to $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$.

Example. Let

$$a(\theta) = \theta^i = e^{i\ln\theta}. \tag{6.5}$$

As the symbol oscillates near the endpoint 0, we examine the behavior of $\gamma_a(\lambda)$ when $\lambda \to +\infty$. Changing the variable $t = 2\lambda\theta$, we have

$$\begin{aligned}
\gamma_a(\lambda) &= \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^\pi \theta^i\, e^{-2\lambda\theta}\, d\theta \\
&= \frac{(2\lambda)^{-i}}{1-e^{-2\pi\lambda}} \int_0^{2\pi\lambda} t\, e^{-t}\, dt \\
&= \frac{(2\lambda)^{-i}}{1-e^{-2\pi\lambda}} \big((1-e^{-2\pi\lambda}) - 2\pi\lambda\, e^{-2\pi\lambda}\big) \\
&= (2\lambda)^{-i}\big(1+o(\lambda)\big),
\end{aligned}$$

where $\lim_{\lambda\to+\infty} o(\lambda) = 0$.

That is, the function $\gamma_a(\lambda)$ oscillates and has no limit when $\lambda \to +\infty$, and thus the Toeplitz operator T_a with symbol (6.5) does not belong to the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$.

The symbol

$$a(\theta) = (\sin\theta)^i$$

provides us with an example for which the corresponding function $\gamma_a(\lambda)$ does not have limits both when $\lambda \to +\infty$ and $\lambda \to -\infty$.

To give a characterization of Toeplitz operators in $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ which have L_∞-symbols we need the following auxiliary result.

Theorem 6.3. *Let $a(\theta) \in L_\infty(0,\pi)$. Then for each real valued monotone function $q(\lambda)$ such that*

$$\lim_{\lambda\to\pm\infty} q(\lambda) = \pm\infty \quad \text{and} \quad \lim_{\lambda\to\pm\infty} \frac{q(\lambda)}{\lambda} = 0 \tag{6.6}$$

we have

$$\lim_{\lambda\to\pm\infty} \Big(\gamma_a\big(\lambda + q(\lambda)\big) - \gamma_a(\lambda)\Big) = 0.$$

Proof. We calculate

$$\begin{aligned}
\gamma_a\big(\lambda+q(\lambda)\big)-\gamma_a(\lambda) &= \frac{2\big(\lambda+q(\lambda)\big)}{1-e^{-2\pi(\lambda+q(\lambda))}}\int_0^\pi a(\theta)e^{-2(\lambda+q(\lambda))\theta}\,d\theta \\
&\quad - \frac{2\lambda}{1-e^{-2\pi\lambda}}\int_0^\pi a(\theta)e^{-2\lambda\theta}\,d\theta \\
&= -\frac{2\big(\lambda+q(\lambda)\big)}{1-e^{-2\pi(\lambda+q(\lambda))}}\int_0^\pi a(\theta)e^{-2\lambda\theta}\big(1-e^{-2q(\lambda)\theta}\big)\,d\theta \\
&\quad + \left(\frac{2\big(\lambda+q(\lambda)\big)}{1-e^{-2\pi(\lambda+q(\lambda))}}-\frac{2\lambda}{1-e^{-2\pi\lambda}}\right)\int_0^\pi a(\theta)e^{-2\lambda\theta}\,d\theta \\
&= I_1(\lambda)+I_2(\lambda)\,.
\end{aligned}$$

Let $\lambda\to+\infty$, we introduce $\sigma(\lambda)=(\lambda\,q(\lambda))^{-1/2}$ and start estimating $I_1(\lambda)$

$$\begin{aligned}
|I_1(\lambda)| &\le \text{const}\left(\lambda\int_0^{\sigma(\lambda)}|a(\theta)|e^{-2\lambda\theta}\big|1-e^{-2q(\lambda)\theta}\big|\,d\theta\right. \\
&\quad \left.+\,\lambda\int_{\sigma(\lambda)}^\pi |a(\theta)|e^{-2\lambda\theta}\,d\theta\right) \\
&\le \text{const}\left(\lambda|q(\lambda)|\int_0^{\sigma(\lambda)}\theta\,e^{-2\lambda\theta}\,d\theta+\lambda\int_{\sigma(\lambda)}^\pi e^{-2\lambda\theta}\,d\theta\right) \\
&\le \text{const}\left(\lambda|q(\lambda)|\left(-\frac{\sigma(\lambda)\,e^{-2\lambda\,\sigma(\lambda)}}{2\lambda}+\frac{1-e^{-2\lambda\,\sigma(\lambda)}}{4\lambda^2}\right)\right. \\
&\quad \left.-\,\lambda\left(\frac{e^{-2\lambda\pi}-e^{-2\lambda\,\sigma(\lambda)}}{2\lambda}\right)\right) \\
&\le \text{const}\left(\left(\frac{q(\lambda)}{\lambda}\right)^{1/2}\cdot e^{-2\left(\frac{\lambda}{q(\lambda)}\right)^{1/2}}+\frac{q(\lambda)}{\lambda}+e^{-2\pi\lambda}+e^{-2\left(\frac{\lambda}{q(\lambda)}\right)^{1/2}}\right).
\end{aligned}$$

By the second condition in (6.6), for each $\varepsilon>0$ and corresponding sufficiently large λ, we have

$$|I_1(\lambda)|<\varepsilon\,.$$

We estimate now $I_2(\lambda)$:

$$\begin{aligned}
|I_2(\lambda)| &\le \text{const}\,|q(\lambda)|\int_0^\pi |a(\theta)|\,e^{-2\lambda\theta}\,d\theta \\
&\le \text{const}\,|q(\lambda)|\int_0^\pi e^{-2\lambda\theta}\,d\theta \le \text{const}\;\frac{q(\lambda)}{\lambda}\,.
\end{aligned}$$

That is, for sufficiently large λ we have as well that

$$|I_2(\lambda)|<\varepsilon\,,$$

and thus

$$\lim_{\lambda\to+\infty}\Big(\gamma_a\big(\lambda+q(\lambda)\big)-\gamma_a(\lambda)\Big)=0\,.$$

The case when $\lambda\to-\infty$ follows from Remark 6.1 and the above arguments. □

Given any $a(\theta)\in L_\infty(0,\pi)$, we introduce now two modified averaging functions which correspond to the endpoints of $[0,\pi]$

$$C'_a(\theta)=\frac{2}{1-e^{-2\theta}}\int_0^\theta a(u)\,du \quad\text{and}\quad D'_a(\theta)=\frac{2}{1-e^{-2\theta}}\int_{\pi-\theta}^\pi a(u)\,du\,. \tag{6.7}$$

We note that these functions are connected with the old averages by

$$C'_a(\theta)=\frac{2}{1-e^{-2\theta}}\,C^{(1)}_a(\theta)\quad\text{and}\quad D'_a(\theta)=\frac{2}{1-e^{-2\theta}}\,D^{(1)}_a(\theta)\,.$$

Both functions $C'_a(\theta)$ and $D'_a(\theta)$ are bounded, moreover $C'_a(\theta)\in C(0,\pi]$ and $D'_a(\theta)\in C[0,\pi)$. That is, to check whether Toeplitz operators with symbols $C'_a(\theta)$ and $D'_a(\theta)$ belong to the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$, one needs to study the behavior of these function near a single point only, 0 for $C'_a(\theta)$ and π for $D'_a(\theta)$.

We changed "... one need ... " to "... one needs ...". Do you agree?

The next theorem shows that the study of general L_∞-symbols is equivalent to the study of these two much more easily treatable functions.

Theorem 6.4. *Let $a(\theta)\in L_\infty(0,\pi)$. Then $\gamma_a(\lambda)\in C(\overline{\mathbb{R}})$ if and only if*

$$\gamma_{C'_a}(\lambda)\in C(\overline{\mathbb{R}})\quad\text{and}\quad\gamma_{D'_a}(\lambda)\in C(\overline{\mathbb{R}})\,. \tag{6.8}$$

Proof. Let $\gamma_a(\lambda)\in C(\overline{\mathbb{R}})$. Consider first the case when $\lambda\to+\infty$. We will integrate by parts in

$$\gamma_{C'_a}(\lambda)=\frac{2\lambda}{1-e^{2\pi\lambda}}\int_0^\pi\left(\frac{2}{1-e^{-2\theta}}\int_0^\theta a(u)\,du\right)e^{-2\lambda\theta}\,d\theta\,.$$

Before doing so we mention that

$$A_\lambda(\theta):=\int_\theta^\infty\frac{e^{-2\lambda u}}{1-e^{-2u}}\,du=\int_\theta^\infty\sum_{n=0}^\infty e^{-2(\lambda+n)u}\,du=\sum_{n=0}^\infty\frac{e^{-2(\lambda+n)\theta}}{2(\lambda+n)}\,.$$

Thus we have

$$\begin{aligned}
\gamma_{C'_a}(\lambda)&=\frac{2\lambda}{1-e^{2\pi\lambda}}\left(\left(-2A_\lambda(\theta)\int_0^\theta a(u)\,du\right)\Bigg|_0^\pi+2\int_0^\pi a(\theta)A_\lambda(\theta)\,d\theta\right)\\
&=-\frac{4\lambda A_\lambda(\pi)}{1-e^{-2\pi\lambda}}\int_0^\pi a(u)\,du+\frac{2\lambda}{1-e^{-2\pi\lambda}}\sum_{n=0}^\infty\frac{1}{\lambda+n}\int_0^\pi a(\theta)\,e^{-2(\lambda+n)\theta}\,d\theta\\
&=-\frac{4\lambda A_\lambda(\pi)}{1-e^{-2\pi\lambda}}\int_0^\pi a(u)\,du+\frac{\lambda}{1-e^{-2\pi\lambda}}\sum_{n=0}^\infty\frac{1-e^{-2\pi(\lambda+n)}}{(\lambda+n)^2}\,\gamma_a(\lambda+n)\,.
\end{aligned}$$

Using the uniform boundedness $|\gamma_a(\lambda+n)| \leq \|\gamma_a\|_{L_\infty(\mathbb{R})}$ and separating the leading term we come to the equality

$$\gamma_{C'_a}(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda}{(\lambda+n)^2}\, \gamma_a(\lambda+n) + o(1)\,. \tag{6.9}$$

It is obvious that

$$\frac{\lambda}{(\lambda+n)^2} = \frac{\lambda}{\lambda+n} - \frac{\lambda}{\lambda+n+1} + \frac{\lambda}{(\lambda+n)^2(\lambda+n+1)}\,.$$

Thus taking into account that

$$\sum_{n=0}^{\infty} \frac{\lambda}{(\lambda+n)^2(\lambda+n+1)} = O\left(\frac{1}{\lambda}\right)$$

we obtain

$$\sum_{n=0}^{\infty} \frac{\lambda}{(\lambda+n)^2} = \left(1 - \frac{\lambda}{\lambda+1} + \frac{\lambda}{\lambda+1} - \frac{\lambda}{\lambda+2} + \frac{\lambda}{\lambda+2} - \cdots\right) + O\left(\frac{1}{\lambda}\right) = 1 + o(1).$$

That is from (6.9) we have

$$\gamma_{C'_a}(\lambda) = \gamma_a(+\infty) + \sum_{n=0}^{\infty} \frac{\lambda}{(\lambda+n)^2}\big(\gamma_a(\lambda+n) - \gamma_a(+\infty)\big) + o(1)\,.$$

As $\gamma_a(\lambda) \in C(\overline{\mathbb{R}})$, for any $\varepsilon > 0$ there is $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$ and each $n \in \mathbb{Z}_+$ we have

$$|\gamma_a(\lambda+n) - \gamma_a(+\infty)| < \varepsilon\,.$$

Thus for $\lambda > \lambda_0$ we have

$$|\gamma_{C'_a}(\lambda) - \gamma_a(+\infty)| < \varepsilon \cdot \sum_{n=0}^{\infty} \frac{\lambda}{(\lambda+n)^2} + o(1) = \varepsilon + o(1)\,,$$

or

$$\lim_{\gamma\to+\infty} \gamma_{C'_a}(\lambda) = \gamma_a(+\infty)\,.$$

The proof that

$$\lim_{\gamma\to-\infty} \gamma_{D'_a}(\lambda) = \gamma_a(-\infty)$$

follows now from Remark 6.1.

Let now $\gamma_{C'_a}(\lambda) \in C(\overline{\mathbb{R}})$. Assuming that $\lambda \to +\infty$, we have

$$\begin{aligned}
\gamma_a(\lambda) &= \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^\pi e^{-2\lambda\theta}\, d\int_0^\theta a(u)\, du \\
&= \frac{2\lambda}{1-e^{-2\pi\lambda}} \left(e^{-2\pi\lambda} \int_0^\pi a(u)\, du + 2\lambda \int_0^\pi \left(\int_0^\theta a(u)\, du \right) e^{-2\lambda\theta}\, d\theta \right) \\
&= \frac{2\lambda}{1-e^{-2\pi\lambda}} \left(\lambda \int_0^\pi C'_a(\theta)(1-e^{-2\theta})\, e^{-2\lambda\theta}\, d\theta \right. \\
&\quad \left. + \frac{1}{2}\, e^{-2\pi\lambda}(1-e^{-2\lambda\pi}) C'_a(\pi) \right) \\
&= \lambda\, \gamma_{C'_a}(\lambda) - \frac{2\lambda^2}{2(\lambda+1)} \cdot \frac{1-e^{-2\pi(\lambda+1)}}{1-e^{-2\pi\lambda}}\, \gamma_{C'_a}(\lambda+1) + O(\lambda e^{-2\pi\lambda}) \\
&= \lambda\, \gamma_{C'_a}(\lambda) - \frac{\lambda^2}{\lambda+1}\, \gamma_{C'_a}(\lambda+1) + O(\lambda e^{-2\pi\lambda})\,.
\end{aligned}$$

That is, we come to the following equality

$$\gamma_{C'_a}(\lambda) - \gamma_{C'_a}(\lambda+1) + \frac{\gamma_{C'_a}(\lambda+1)}{\lambda+1} + O(e^{-2\pi\lambda}) = \frac{\gamma_a(\lambda)}{\lambda}\,. \tag{6.10}$$

Changing, if necessary, the initial symbol $a(\theta)$ by adding a constant, we may assume without loss of generality that

$$\lim_{\lambda\to+\infty} \gamma_{C'_a}(\lambda) = 0\,.$$

Introduce the function

$$\alpha(\lambda) = \sup_{\xi\geq\lambda} |\gamma_{C'_a}(\xi)|\,,$$

which is non increasing and satisfies

$$\lim_{\lambda\to+\infty} \alpha(\lambda) = 0\,.$$

Substitute

$$\lambda+1\,, \quad \lambda+2\,, \quad \ldots\,, \quad \lambda + \left[\lambda\cdot\alpha^{1/2}(\lambda)\right],$$

for λ in (6.10), where $[\,\cdot\,]$ is the entire part of a number; summing up the obtained equalities, we have

$$\begin{aligned}
\gamma_{C'_a}(\lambda) - \gamma_{C'_a}(\lambda+n_0+1) + \sum_{n=0}^{n_0} \frac{\gamma_{C'_a}(\lambda+n+1)}{\lambda+n+1} + O(n_0 e^{-2\pi\lambda}) \\
= \sum_{n=0}^{n_0} \frac{\gamma_a(\lambda+n)}{\lambda+n}\,,
\end{aligned} \tag{6.11}$$

where $n_0 = n_0(\lambda) = [\lambda\cdot\alpha^{1/2}(\lambda)]$.

We assume now that

$$\lim_{\lambda\to+\infty} \gamma_a(\lambda) \neq 0\,.$$

That is, there exists a sequence $\{\lambda_k\}_{k=1}^{\infty}$ which tends to $+\infty$ and such that for some $\sigma > 0$

$$|\gamma_a(\lambda_k)| \geq \sigma\,, \quad \text{for all} \quad k = 1, 2, \ldots\,.$$

We denote by $E_1(\lambda)$ the left hand side of the equality in (6.11) and estimate it:

$$\begin{aligned}
|E_1(\lambda_k)| &\leq 2\alpha(\lambda_k) + \alpha(\lambda_k) \sum_{n=0}^{n_0} \frac{1}{\lambda_k + n + 1} + O\big((n_0+1)e^{-2\pi\lambda_k}\big) \\
&\leq 2\alpha(\lambda_k) + \alpha(\lambda_k)\, \ln \frac{\lambda_k + n_0 + 1}{\lambda_k + 1} + O\big((n_0+1)e^{-2\pi\lambda_k}\big) \\
&= 2\alpha(\lambda_k) + \alpha(\lambda_k)\, \ln \left(1 + \frac{n_0}{\lambda_k + 1}\right) + O\big((n_0+1)e^{-2\pi\lambda_k}\big) \\
&\leq \text{const} \left(\alpha(\lambda_k)\big(1 + \alpha^{1/2}(\lambda_k)\big) + O\big((n_0+1)e^{-2\pi\lambda_k}\big)\right) \\
&\leq \text{const} \left(\alpha(\lambda_k) + O\big((n_0+1)e^{-2\pi\lambda_k}\big)\right).
\end{aligned}$$

We denote now the right hand side of the equality in (6.11) by $E_2(\lambda)$ and estimate it:

$$\begin{aligned}
E_2(\lambda_k) &= \sum_{n=0}^{n_0} \frac{\gamma_a(\lambda_k)}{\lambda_k + n} + \sum_{n=0}^{n_0} \frac{\gamma_a(\lambda_k + n) - \gamma_a(\lambda_k)}{\lambda_k + n} \\
&= \gamma_a(\lambda_k)\, \ln \frac{\lambda_k + n_0}{\lambda_k} + o(1) + E_{2,2}(\lambda_k)\,,
\end{aligned}$$

where

$$E_{2,2}(\lambda_k) = \sum_{n=0}^{n_0} \frac{\gamma_a(\lambda_k + n) - \gamma_a(\lambda_k)}{\lambda_k + n}\,.$$

As the function $n_0 = n_0(\lambda)$ satisfies (6.6), we make use of Theorem 6.3. That is, for each $k \in \mathbb{N}$ there is $\sigma_k > 0$ such that for each $n \in [1, n_0] \cap \mathbb{N}$

$$|\gamma_a(\lambda_k) - \gamma_a(\lambda_k + n)| < \sigma_k \quad \text{and} \quad \lim_{k\to\infty} \sigma_k = 0\,.$$

We have

$$|E_{2,2}(\lambda_k)| \leq \sigma_k\, \ln \frac{\lambda_k + n_0(\lambda_k)}{\lambda_k} \leq \frac{n_0(\lambda_k)}{\lambda_k}\,,$$

and thus

$$\left| E_2(\lambda_k) - \gamma_a(\lambda_k) \cdot \frac{n_0(\lambda_k)}{\lambda_k} \right| \leq O\left(\left(\frac{n_0(\lambda_k)}{\lambda_k}\right)^2 + \sigma_k \frac{n_0(\lambda_k)}{\lambda_k} \right).$$

This yields

$$|E_2(\lambda_k)| \geq \frac{|\gamma_a(\lambda_k)|}{2} \cdot \frac{n_0(\lambda_k)}{\lambda_k} \geq \frac{|\gamma_a(\lambda_k)|}{2}\, \alpha^{1/2}(\lambda_k)\,,$$

or

$$|E_2(\lambda_k)| \geq \frac{\sigma}{4}\, \alpha^{1/2}(\lambda_k)\,.$$

Substituting $E_1(\lambda_k)$ and $E_2(\lambda_k)$ in (6.11) by their estimates we have

$$\mathrm{const}\big(\alpha(\lambda_k) + \lambda_k\, \alpha^{1/2}(\lambda_k)\, e^{-2\pi\lambda_k} + e^{-2\pi\lambda_k}\big) \geq \frac{\sigma}{4}\, \alpha^{1/2}(\lambda_k)\,.$$

Now, if

$$\lim_{k\to\infty} \frac{e^{-2\pi\lambda_k}}{\alpha^{1/2}(\lambda_k)} = 0\,, \tag{6.12}$$

then we come to a contradiction, and thus

$$\lim_{\lambda\to+\infty} \gamma_a(\lambda) = 0\,.$$

If (6.12) does not hold then there exist $\sigma_1 > 0$ and a subsequence $\{\lambda_{k_l}\}_{l=1}^{\infty}$ of the sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that for each $l \in \mathbb{N}$

$$e^{-2\pi\lambda_{k_l}} \geq \sigma_1\, \alpha^{1/2}(\lambda_{k_l})\,.$$

Then substituting $\lambda = \lambda_{k_l}$ in (6.10) we come to

$$\frac{e^{-4\pi\lambda_{k_l}}}{\sigma_1^2} + \frac{e^{-4\pi\lambda_{k_l}}}{\sigma_1^2}\left(1 + \frac{1}{\lambda_{k_l}+1}\right) + \mathrm{const}\, e^{-2\pi\lambda_{k_l}} \geq \frac{\sigma}{\lambda_{k_l}}\,,$$

which again leads to a contradiction.

The proof that $\gamma_a(\lambda)$ is continuous at the point $-\infty$ again follows from Remark 6.1. □

As a final remark we note that the above results uncover a variety of Toeplitz operators in $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ whose bounded symbols belong to a more general class than one of Proposition 5.5, extending thereby the descriptions of Toeplitz operators in $\mathcal{T}(PC(\overline{\mathbb{D}},T))$ of Section 5.

At the same time we shown that the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ contains as well bounded Toeplitz operators with *unbounded* symbols. The detailed exploration of this interesting phenomenon and of its impact to Toeplitz operators in $\mathcal{T}(PC(\overline{\mathbb{D}},T))$ we leave for a further study.

Do you mean "... we have shown ..." instead of "... we shown ..."?

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