Fox-Li operator, Laser Theory and Wiener-Hopf Theory

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This talk is based on joint work with
Albrecht Böttcher and Arieh Iserles.

Albrecht Böttcher, Sergei Grudsky and Arieh Iserles. *Spectral theory of large Wiener-Hopf operators with complex- symmetric kernels and rational symbols.* Mathematical Proceedings of Cambridge Philosophical Society, 151 (2011), 161-191 pp.

A truncated Wiener–Hopf operator is of the form

$$(K_\tau f)(t) := f(t) + \int_0^\tau \mathrm{k}(t-s)f(s)ds, \qquad t \in (0,\tau). \tag{1}$$

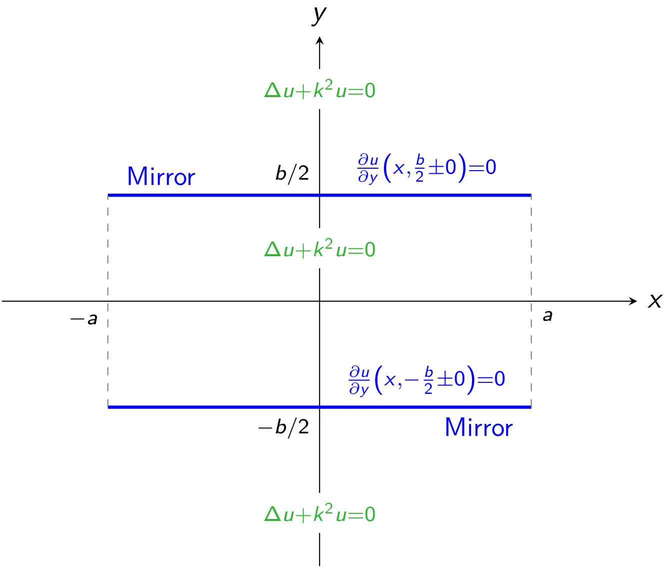
We suppose that k is a function in $L^2(\mathbb{R})$, so that the integral operator in (1) is a Hilbert–Schmidt operator and thus compact on $L^2(0,\tau)$ for all $\tau > 0$. Let $\mathrm{sp}\, K_\tau$ be the spectrum of K_τ. Since $K_\tau - I$ is compact, all points in $\mathrm{sp}\, K_\tau \setminus \{1\}$ are eigenvalues. We are interested in the location and the asymptotic behaviour of these eigenvalues as τ tends to infinity.

The basic assumptions stipulated in this reports are that the kernel $\mathrm{k}(t-s)$ is complex-symmetric, which means that k is a complex–valued function satisfying $\mathrm{k}(t) = \mathrm{k}(-t)$ for all $t \in \mathbb{R}$, and that the so-called *symbol* of the operator,

$$a(x) := 1 + \int_{-\infty}^{\infty} \mathrm{k}(t)\, e^{\mathrm{i}xt} dt, \quad x \in \mathbb{R},$$

is complex-symmetric also.

A.G. Fox and T. Li *Resonant modes in a Maser Interferometer.* Bell, System Tech. J 40, 1961.



The problem about the

PROPER WAVES (RESONANT MODES)

of this waveguide.

$$\int_{-a}^{a} k(x-s)u(s)ds = \lambda u(x), \quad x \in (-a, a)$$

$$a(\mu) = (\Phi K)(\mu) = k\frac{(1 - e^{2ib\sqrt{k^2-\mu^2}})}{\sqrt{k^2 - \mu^2}}$$

$k = w/c$- wave number.

If $|\mu| \ll |k|$, then

$$b\sqrt{k^2 - \mu^2} = kb\sqrt{1 - (\mu/k)^2} \approx bk(1 - (\mu/k)^2 + \frac{3}{8}((\mu/k)^4)$$

If $b|k^{-3}|\,|\mu|^4 \ll 1$, then

$$\sqrt{k^2 - \mu^2} \approx k - \mu^2/k$$

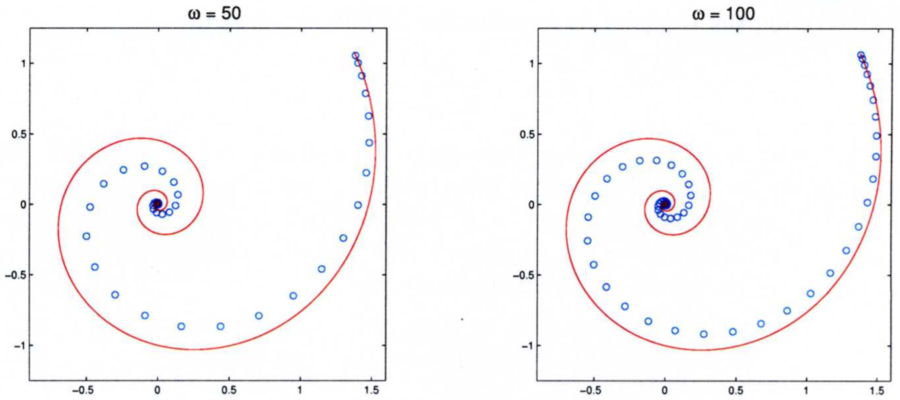
and

$$a^*(\mu) = (1 - e^{ikb - i\frac{b}{k}\mu^2})$$

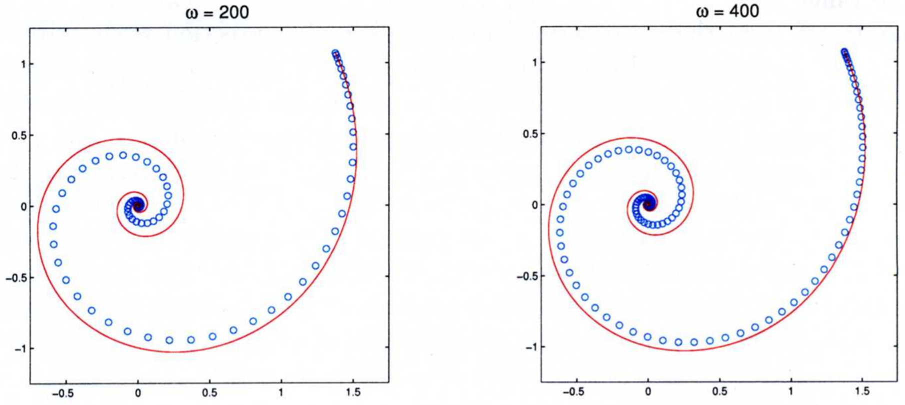
$$a_0(\mu) = e^{-iw\mu^2} <=> k(x) = e^{iwx^2}$$

Strongly oscillating symbol $<=>$ kernel.

A. Böttcher, H. Brunner, A. Iserles, and S. Nørsett, *On the singular values and eigenvalues of the Fox–Li and related operators.* New York J. Math., to appear.



A. Böttcher, H. Brunner, A. Iserles, and S. Nørsett, *On the singular values and eigenvalues of the Fox–Li and related operators.* New York J. Math., to appear.



The are a lot articles (numerical) that devoted to the case $a_0(\mu) = e^{iw\mu^2}$. But there exist very few rigorous mathematical results.
The change $a(t)$ by $a^*(t)$.

1. The spectrum (general speaking) in not stable under (even small) perturbation.
2. The symbol $a_0(\mu)$ is strongly oscillation including the point $\mu \to \infty$. The symbol
$$a(\mu) = k \frac{(1 - e^{2ib\sqrt{k^2-\mu^2}})}{\sqrt{k^2 - \mu^2}}$$
is strongly oscillating only for $|\mu| \ll k$ and continuous in $\mu \to \infty$
$$a(\infty) = 0$$
That is the symbol $a(\mu)$ in point of view Wiener-Hopf operator theory is more simple.

Böttcher/Widom 1994.
If $K(t)$ is complex-symmetric and $a(\mu) \in C(\dot{R})$ then

$$\lim_{\tau\to\infty} \mathrm{sp}\; K_\tau = \mathrm{Im}\, a(\mu), \quad (\mu \in R)$$

Asymptotics of eigenvalues by

$$\tau \to \infty\,?$$

Let $k(t)$ is complex-symmetric and $a(\mu)$ is rational, then

$$\mathrm{k}(t) = \begin{cases} \sum_{\ell=1}^{m} p_\ell(t) \mathrm{e}^{-\lambda_\ell t} & \text{for} \quad t > 0, \\ \sum_{\ell=1}^{m} p_\ell(-t) \mathrm{e}^{\lambda_\ell t} & \text{for} \quad t < 0, \end{cases}$$

where λ_ℓ are complex numbers with $\mathrm{Re}\,\lambda_\ell > 0$ and $p_\ell(t)$ are polynomials with complex coefficients. As $\mathrm{k}(t) = \mathrm{k}(-t)$ for all $t \in \mathbb{R}$ if and only if $a(x) = a(-x)$ for all $x \in \mathbb{R}$, the Wiener–Hopf operators considered here are just those with even rational symbols. Moreover, $\mathrm{k} \in L_2(\mathbb{R})$ implies that $\lim_{|x| \to \infty} a(x) = 1$. Therefore we may write

$$a(x) = \prod_{j=1}^{r} \frac{x^2 - \zeta_j^2}{x^2 + \mu_j^2}, \quad x \in \mathbb{R}, \tag{2}$$

where $\zeta_j \in \mathbb{C}$, $\mu_j \in \mathbb{C}$, $\mathrm{Re}\,\mu_j > 0$, and $-\zeta_j^2 \neq \mu_k^2$ for all j, k. To indicate the dependence of K_τ on the symbol a and in accordance with the literature, we henceforth denote K_τ by $W_\tau(a)$.

We provide here asymptotic expansions for individual eigenvalues. Under additional hypotheses, namely that the set $\mathcal{R}(a)$ is a curve without self-intersections and that the roots of certain polynomials are all simple, we prove the following. We associate a number $\beta > 0$ with a, consider the half-stripe

$$S_\tau := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0, \quad |\mathrm{Im}\, z| \le \beta/\tau\},$$

and show that, for τ large enough, all the eigenvalues of $W_\tau(a)$ are contained in $a(S_\tau)$.
We consider the segments

$$I_{k,\tau} := \left[\left(k - \frac{1}{2}\right)\frac{\pi}{\tau}, \left(k + \frac{1}{2}\right)\frac{\pi}{\tau}\right]$$

In this way we obtain family of rectangles

$$S_{k,\tau} := \{z \in S_\tau : \mathrm{Re}\, z \in I_{k,\tau}\}.$$

We prove that if τ is sufficiently large then each set $a(S_{k,\tau})$ contains exactly one eigenvalue, and the eigenvalue $\lambda_{k,\tau}$ in $a(S_{k,\tau})$ has an asymptotic expansion

$$\lambda_{k,\tau} \sim a(k\pi/\tau) + \frac{c_1(k\pi/\tau)}{2\mathrm{i}\tau} + \frac{c_2(k\pi/\tau)}{(2\mathrm{i}\tau)^2} + \dots$$

with computable coefficients $c_1(k\pi/\tau), c_2(k\pi/\tau), \dots$. We also show that eigensubspaces are all one-dimensional and describe the structure of the eigenfunctions.

The Wiener–Hopf determinant

Let $U \subset \mathbb{C}$ be a sufficiently small open neighbourhood of $\mathcal{R}(a)$ and take a point $\lambda \in U \setminus \{a(0), 1\}$ such that the roots $\omega_2(\lambda), \ldots, \omega_r(\lambda)$ are all distinct. We then have

$$\frac{a(x) - \lambda}{1 - \lambda} = \frac{(x - \xi_1(\lambda)) \ldots (x - \xi_{2r}(\lambda))}{(x^2 + \mu_1^2) \ldots (x^2 + \mu_r^2)} = \prod_{j=1}^{r} \frac{x^2 - \omega_j(\lambda)^2}{x^2 + \mu_j^2}.$$

Thus, $\xi_1(\lambda), \ldots, \xi_{2r}(\lambda)$ are simply the roots $\pm\omega_1(\lambda), \ldots, \pm\omega_r(\lambda)$ labelled in a different manner.

A. Böttcher, 1989:

$$\det W_\tau\left(\frac{a-\lambda}{1-\lambda}\right) = \mathrm{e}^{\kappa\tau}\sum_M W_M \mathrm{e}^{w_M\tau} \tag{3}$$

where $\kappa = \kappa(\lambda)$ is some constant, the sum is over all subsets $M \subset \{\xi_1, \ldots, \xi_{2r}\}$ of cardinality r, and, with $M^c := \{\xi_1, \ldots, \xi_{2r}\} \setminus M$ and $R := \{\mu_1, \ldots, \mu_r\}$,

$$\begin{aligned} w_M &:= \sum_{\xi_j \in M^c} \mathrm{i}\xi_j, \\ W_M &:= \frac{\prod_{\xi_j \in M^c, \mu_m \in R}(\mathrm{i}\xi_j + \mu_m)\prod_{\mu_\ell \in R, \xi_k \in M}(\mu_\ell - \mathrm{i}\xi_k)}{\prod_{\mu_\ell \in R, \mu_m \in R}(\mu_\ell + \mu_m)\prod_{\xi_j \in M^c, \xi_k \in M}(\mathrm{i}\xi_j - \mathrm{i}\xi_k)}. \end{aligned}$$

The point λ belongs to $\mathrm{sp}\, W_\tau(a)$ if and only if (3) is zero, whereby its algebraic multiplicity is its multiplicity as a zero of (3).

The dominant terms in (3) are those for which

$$\operatorname{Im} w_M = \sum_{\xi_j \in M^c} \operatorname{Im} \xi_j \tag{4}$$

is minimal.
The two candidates for sets M with minimal values (4) are given by

$$M_1^c := \{-\omega_1, -\omega_2, \ldots, -\omega_r\}, \quad M_2^c := \{\omega_1, -\omega_2, \ldots, -\omega_r\},$$

$$\mathrm{e}^{-\kappa\tau} \det W_\tau \left(\frac{a-\lambda}{1-\lambda}\right) = W_{M_1} \mathrm{e}^{w_{M_1}\tau} + W_{M_2} \mathrm{e}^{w_{M_2}\tau} + \sum_{M \neq M_1, M_2} W_M \mathrm{e}^{w_M \tau}, \tag{5}$$

Fix an open neighborhood $U \subset \mathbb{C}$ of $\mathcal{R}(a)$. Then $\operatorname{sp} W_\tau(a) \subset U$ for all sufficiently large τ. Let $\Pi = \{z \in \mathbb{C} : |\operatorname{Im} z| < \delta, a(z) \in U\}$. For $z \in \Pi$ consider the two functions

$$Q(z) := \prod_{\ell=1}^{r} (z - \mathrm{i}\mu_\ell),$$

$$P(z) := \prod_{\ell=2}^{r} [z - \omega_\ell(a(z))]$$

and set

$$b(z) := \frac{Q(-z)^2}{Q(z)^2} \cdot \frac{P(z)^2}{P(-z)^2}.$$

It be $\lambda = a(z)$ with $z = \omega_1 = \omega_1(\lambda)$ in Π. The equation $\det W_\tau((a-\lambda)/(1-\lambda)) = 0$ may be written in the form

$$e^{2i\tau z} = b(z)(1 + \varphi_\tau(z)) \tag{6}$$

where

$$\varphi_\tau(z) = \sum_{M \neq M_1, M_2} W_{M_1}^{-1} W_M e^{(w_M - w_{M_1})\tau}.$$

Lemma

If $0 \notin \Pi$ *or if* $0 \in \Pi$ *but the roots* $\omega_2(a(0)), \ldots, \omega_r(a(0))$ *are distinct, then*

$$\varphi_\tau(z) = O(\mathrm{e}^{-2\delta\tau}) \quad \textit{and} \quad \varphi'_\tau(z) = O(\tau \mathrm{e}^{-2\delta\tau})$$

uniformly in $z \in \Pi$.

Main results

Theorem (1)

Let $\operatorname{clos} I$ *be the closure of* I *in* $[0, \infty]$ *and suppose that for* λ *in* $a(\operatorname{clos} I)$ *the roots* $\omega_2(\lambda), \ldots, \omega_r(\lambda)$ *are distinct. Then there exists a* τ_0 *such that the following is true for every* $\tau > \tau_0$.

(a) *If* $\lambda = a(z) \in U$ *is an eigenvalue of* $W_\tau(a)$ *such that* $\operatorname{Re} z \in I_{k,\tau}$ *for some* $k \in \mathcal{K}_\tau(I)$, *then* $z \in S_{k,\tau}$.

(b) *For each* $k \in \mathcal{K}_\tau(I)$, *the set* $a(S_{k,\tau})$ *contains exactly one eigenvalue* $\lambda_{k,\tau}$ *of the operator* $W_\tau(a)$. *The algebraic multiplicity of this eigenvalue is* 1.

Theorem (1)

(c) *The function*

$$\Phi_{k,\tau}(z) := \frac{k\pi}{\tau} + \frac{1}{2\mathrm{i}\tau}\log b(z)$$

is a contractive map of $S_{k,\tau}$ into itself and, letting

$$z_{k,\tau}^{(0)} := \frac{k\pi}{\tau}, \quad z_{k,\tau}^{(n)} := \Phi_{k,\tau}(z_{k,\tau}^{(n-1)}) \qquad (n \geq 1),$$

we have

$$\lambda_{k,\tau} = a(z_{k,\tau}^{(n)}) + O(1/\tau^{n+1}) \ \text{as } \tau \to \infty$$

uniformly in $k \in \mathcal{K}_\tau(I)$, that is, there exist constants $C_n < \infty$ independent of k and τ such that

$$|\lambda_{k,\tau} - a(z_{k,\tau}^{(n)})| \leq C_n/\tau^{n+1}$$

for all $\tau > \tau_0$ and all $k \in \mathcal{K}_\tau(I)$.

Corollary

If the points $\omega_2(1), \ldots, \omega_r(1)$ are distinct then $W_\tau(a)$ has infinitely many eigenvalues for every sufficiently large τ.

The first three iterations in Theorem 1(c) give for $\lambda_{k,\tau}$

$$a(z_0) + \frac{1}{2\mathrm{i}\tau} a'(z_0)c_1(z_0) + \frac{1}{(2\mathrm{i}\tau)^2}\left[a'(z_0)c_2(z_0) + \frac{a''(z_0)}{2}c_1(z_0)^2\right] + \frac{1}{(2\mathrm{i}\tau)^3}\left[a'(z_0)c_3(z_0) + a''(z_0)c_1(z_0)c_2(z_0) + \frac{a'''(z_0)}{6}c_1(z_0)^3\right] + O\left(\frac{1}{\tau^4}\right),$$

where

$$c_1(z_0) = \log b(z_0), \qquad c_2(z_0) = \frac{b'(z_0)}{b(z_0)}\log b(z_0),$$

$$c_3(z_0) = \frac{b'(z_0)^2}{b(z_0)^2}\log b(z_0) + \frac{b''(z_0)b(z_0) - b'(z_0)^2}{2b(z_0)^2}(\log b(z_0))^2.$$

If a is real valued, which occurs if and only if $\mathrm{k}(t) = \overline{\mathrm{k}(-t)}$ for all t, then $W_\tau(a)$ is a selfadjoint operator. In this case $|b(x)| = 1$ for $x \in \mathbb{R}$, hence the function $\Phi_{k,\tau}$ in Theorem 1(c) maps $I_{k,\tau}$ into itself and becomes

$$\Phi_{k,\tau}(x) = \frac{k\pi}{\tau} + \frac{1}{2\tau} \arg b(x)$$

for $x \in I_{k,\tau}$. It follows in particular that all eigenvalues are real, as they should be for a selfadjoint operator.

Eigenfunctions

Theorem (2)

Suppose that the numbers $\mu_1, \ldots, \mu_r$ are distinct. Let λ be an eigenvalue of $W_\tau(a)$ and assume that the roots $\omega_2(\lambda), \ldots, \omega_r(\lambda)$ are distinct. Then every eigenfunction $\varphi_\tau \in L^2(0, \tau)$ of $W_\tau(a)$ corresponding to λ is of the form

$$\varphi_\tau(t) = \sum_{j=1}^{r} \left[c_j \mathrm{e}^{\mathrm{i}\omega_j(\lambda)t} + c_{r+j} \mathrm{e}^{-\mathrm{i}\omega_j(\lambda)t} \right], \tag{7}$$

satisfies $\varphi_\tau(\tau - t) = \theta \varphi_\tau(t)$ for all $t \in (0, \tau)$ with $\theta \in \{\pm 1\}$,

Theorem (2)

and can be rewritten in the form

$$\varphi_\tau(t) = \begin{cases} \sum_{j=1}^{r} 2c_j e^{i\omega_j(\lambda)\tau/2} \cos\left(\omega_j(\lambda)\left(t - \frac{\tau}{2}\right)\right) & \text{for} \quad \theta = 1, \\ \sum_{j=1}^{r} 2ic_j e^{i\omega_j(\lambda)\tau/2} \sin\left(\omega_j(\lambda)\left(t - \frac{\tau}{2}\right)\right) & \text{for} \quad \theta = -1. \end{cases}$$

The coefficients c_j can be computed from the linear algebraic system.

Numerical examples

$$a(x) = \frac{-(16+68\mathrm{i}) - (10+30\mathrm{i})x^2 - (3+2\mathrm{i})x^2 + x^6}{(12+16\mathrm{i}) + (20+12\mathrm{i})x^2 + (9-4\mathrm{i})x^4 + x^6} = 1 + 2\sum_{k=1}^{3} \frac{\alpha_k \mu_k}{x^2+\mu_k^2} \qquad (8)$$

where $\boldsymbol{\alpha} = [-1, -\mathrm{i}, -2]$ and $\boldsymbol{\mu} = [1, 1+\mathrm{i}, 3-\mathrm{i}]$.

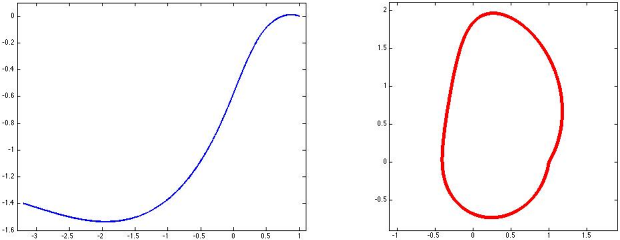


Figure: The range $\mathcal{R}(a)$ is indicated on the left, while the range of b on $(0, \infty)$ is indicated on the right. The latter is traced out clockwise, starting and terminating at 1.

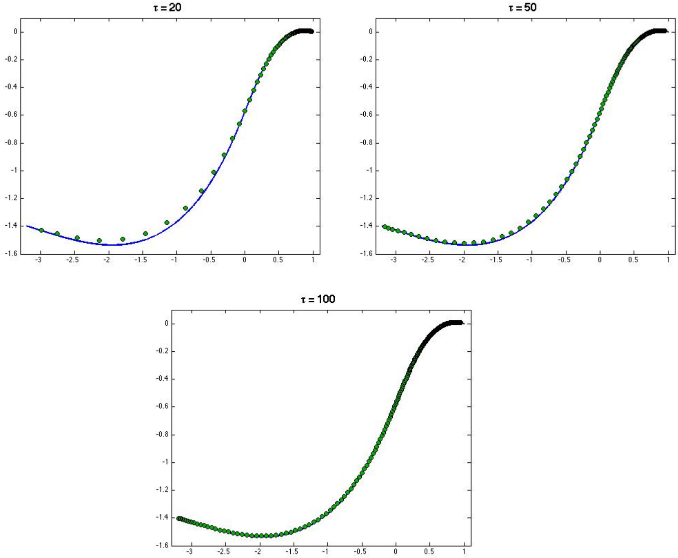


Figure: The eigenvalues, denoted by small discs and overlaid on $\mathcal{R}(a)$, for $\tau = 20, 50, 100$.

Speed of convergence

Table: The error $|z_{k,\tau}^{(n)} - \lambda_{k,\tau}|$ for $\tau = 20$, $k = 10, \ldots, 17$ and the iterations $n = 0, 1, 2, 3, 4$.

	10	11	12	13	14	15	17
0	9.84_{-02}	7.69_{-02}	6.02_{-02}	4.75_{-02}	3.80_{-02}	3.10_{-02}	2.19_{-02}
1	3.69_{-03}	2.68_{-03}	1.95_{-03}	1.42_{-03}	1.03_{-03}	7.55_{-04}	4.11_{-04}
2	1.40_{-04}	9.47_{-05}	6.38_{-05}	4.26_{-05}	2.82_{-05}	1.84_{-05}	7.70_{-06}
3	5.33_{-06}	3.55_{-06}	2.09_{-06}	1.28_{-06}	7.70_{-07}	4.51_{-07}	1.44_{-07}
4	2.03_{-07}	1.18_{-07}	6.85_{-08}	3.86_{-08}	2.10_{-08}	1.10_{-08}	2.71_{-09}

Table: The error $|z_{k,\tau}^{(n)} - \lambda_{k,\tau}|$ for $\tau = 100$, $k = 50, 55, \ldots, 85$ and the iterations $n = 0, 1, 2, 3, 4$.

	50	55	60	65	70	75	85
0	2.09_{-02}	1.62_{-02}	1.26_{-02}	9.87_{-03}	7.86_{-03}	6.38_{-03}	4.48_{-03}
1	1.66_{-04}	1.19_{-04}	8.50_{-05}	6.11_{-05}	4.42_{-05}	3.21_{-05}	1.73_{-05}
2	1.32_{-06}	8.69_{-07}	5.75_{-07}	3.79_{-07}	2.48_{-07}	1.62_{-07}	6.72_{-08}
3	1.05_{-08}	6.37_{-09}	3.89_{-09}	2.35_{-09}	1.40_{-09}	8.14_{-10}	2.60_{-10}
4	8.32_{-11}	4.67_{-11}	2.63_{-11}	1.46_{-11}	7.86_{-12}	4.10_{-12}	1.01_{-12}

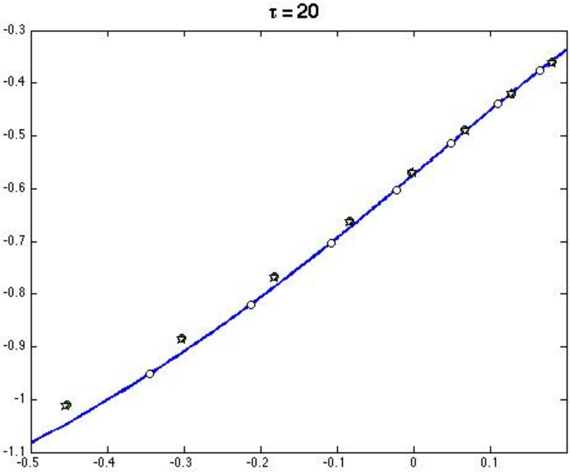


Figure: The speed of convergence of $a^{(n)}_{k,20}$ for growing n. The equispaced points $z^{(0)}_{k,20}$ are denoted by white circles, the first iteration $z^{(1)}_{k,20}$ by filled-in discs and the eigenvalues $\lambda^{(n)}_{k,20}$ by white stars.

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