Uniform individual asymptotics for the eigenvalues and eigenvectors of large Toeplitz matrices

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The International Workshop
WIENER-HOPF METHOD, TOEPLITZ OPERATORS, AND THEIR APPLICATIONS
Boca del Rio, Veracruz, Mexico
November 03-07, 2015

Abstract

The asymptotic behavior of the spectrum of large Toeplitz matrices has been studied for almost one century now. Among this huge work, we can find the Szegö theorems on the eigenvalue distribution and the asymptotics for the determinants, as well as other theorems about the individual asymptotics for the smallest and largest eigenvalues.

Results about uniform individual asymptotics for all the eigenvalues and eigenvectors appeared only five years ago. The goal of the present lecture is to review this area, to talk about the obtained results, and to discuss some open problems.

This review is based on joint works with Manuel Bogoya, Albrecht Böttcher, and Egor Maximenko.

Main object.

Spectral properties of larger finite Toeplitz matrices

$$A_n = (a_{j-k})_{j,k=0}^{n-1} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \dots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \dots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix}.$$

$a(t) = \sum_{j=-\infty}^{\infty} a_j t^j, \; t \in \mathbb{T}$-symbol of $\{A_n\}_{n=1}^{\infty}$

Eigenvalues, eigenvectors singular values, condition numbers, invertibility and norms of inverses, e.t.c.

$n \sim 1000$ is a business of numerical linear algebra.

Statistical physics - $n = 10^7 - 10^{12}$ - is a business of asymptotic theory.

I. Two parameters:

n- dimensions of matrices;

j- number of eigenvalue

$$1 \leq j \leq n$$

Asymptotics by n uniformly in j.

II. Distance between λ_j and λ_{j+1} is small:

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n}\right) \text{ -- normal case}$$

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n^\gamma}\right) \text{ -- special case}$$

$$\lambda_j = \lambda_{j+1} \qquad \text{– exceptional case}$$

Publications about asymptotics of individual eigenvalues and eigenvectors.

1. Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Inside the Eigenvalues of Certian of Hermitian Toeplitz Band Matrices. Computational and Applied Mathematics, 233 (2010), 2245-2264 pp.

2. Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. On the Structure of the Eigenvectors of Large Hermitian Toeplitz Band Matrices. Operator Theory: Advances and Applications, 210 (2010), 15-36 pp.

3. Deift P, Its A, and Krasovsky I. Eigenvalues of Toeplitz matrices in the bulk of the spectrum. Bulletin of the Institute of Mathematics Academia Sinica (New Series) 7 (2012), 437-461 pp.

4. J. M. Bogoya, Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Eigenvalues of Hermitian Toeplitz matrices with smooth simple-loop symbols. Journal of Mathematical Analysis and Applications Volume 422, Issue 2, 15 February 2015, 1308-1334 pp.

5. H. Dai, Z. Geary, L.P. Kadanoff. Asymptotics of eigenvalues and eigenvectors of Toeplitz matrices. Journal of Statistical Mechanics: Theory and Experiment, May, 2009, PO5012.

6. J. M. Bogoya, Albrecht Böttcher and Sergei M. Grudsky. Asymptotics of individual eigenvalues of a class of large Hessenberg Toeplitz matrices. Operator Theory: Advances and Applications, 220 (2012), 77-95 pp.

7. J. M. Bogoya, Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Eigenvectors of Hessenberg Toeplitz matrices and a problem by Dai Geary and Kadanoff. Linea Algebra and its Applications, 436 (2012), 3480-3492 pp.

8. Albrecht Böttcher, Sergei Grudsky and Arieh Iserles. Spectral theory of large Wiener-Hopf operators with complex-symmetric kernels and rational symbols. Mathematical Proceedings of Cambridge Philosophical Society, 151 (2011), 161-191 pp.

Main results-simple loop case

For $\alpha \geq 0$, we denote by W^α the weighted Wiener algebra of all functions $a : \mathbb{T} \to \mathbb{C}$ whose Fourier coefficients satisfy

$$\|a\|_\alpha := \sum_{j=-\infty}^{\infty} |a_j|(|j|+1)^\alpha < \infty.$$

Let m be the entire part of α. It is readily seen that if $a \in W^\alpha$ then the function g defined by $g(\sigma) := a(e^{i\sigma})$ is a 2π-periodic C^m function on $\mathbb{R}$. In what follows we consider real-valued simple-loop functions in W^α. To be more precise, for $\alpha \geq 2$, we let SL^α denote the set of all $a \in W^\alpha$ such that g has the following properties: the range of g is a segment $[0, M]$ with $M > 0$, $g(0) = g(2\pi) = 0$, $g''(0) = g''(2\pi) > 0$, and there is a $\varphi_0 \in (0, 2\pi)$ such that $g(\varphi_0) = M$, $g'(\sigma) > 0$ for $\sigma \in (0, \varphi_0)$, $g'(\sigma) < 0$ for $\sigma \in (\varphi_0, 2\pi)$, and $g''(\varphi_0) < 0$.

Let $a \in SL^\alpha$. Then for each $\lambda \in [0, M]$, there are exactly one $\varphi_1(\lambda) \in [0, \varphi_0]$ such that $g(\varphi_1(\lambda)) = \lambda$ and exactly one $\varphi_2(\lambda) \in [\varphi_0, 2\pi]$ satisfying $g(\varphi_2(\lambda)) = \lambda$. For each $\lambda \in [0, M]$, the function g takes values less than or equal to λ on the segments $[0, \varphi_1(\lambda)]$ and $[\varphi_2(\lambda), 2\pi]$. Denote by $\varphi(\lambda)$ the arithmetic mean of the lengths of these two segments,

$$\varphi(\lambda) := \frac{1}{2}(\varphi_1(\lambda) - \varphi_2(\lambda)) + \pi = \frac{1}{2}\mu\{\sigma \in [0, 2\pi] : \ g(\sigma) \le \lambda\},$$

where μ is the Lebesgue measure on $[0, 2\pi]$. The function $\varphi : [0, M] \to [0, \pi]$ is continuous and bijective. We let $\psi : [0, \pi] \to [0, M]$ stand for the inverse function.

Put

$$\sigma_1(s) = \varphi_1(\psi(s)) \text{ and } \sigma_2(s) = \varphi_2(\psi(s)).$$

Then

$$g(\sigma_1(s)) = g(\sigma_2(s)) = \psi(s).$$

Let further

$$\beta(\sigma, s) := \frac{(g(\sigma) - \psi(s))e^{is}}{(e^{i\sigma} - e^{i\sigma_1(s)})(e^{-i\sigma} - e^{-i\sigma_2(s)})} = \frac{\psi(s) - g(\sigma)}{4\sin\frac{\sigma-\sigma_1(s)}{2}\sin\frac{\sigma-\sigma_2(s)}{2}}.$$

We will show that β is a continuous and positive function on $[0, 2\pi] \times [0, \pi]$. We define the function $\eta : [0, \pi] \to \mathbb{R}$ by

$$\eta(s) := \theta(\psi(s)) = \frac{1}{4\pi}\int_0^{2\pi} \frac{\log\beta(\sigma, s)}{\tan\frac{\sigma-\sigma_2(s)}{2}}\, d\sigma - \frac{1}{4\pi}\int_0^{2\pi} \frac{\log\beta(\sigma, s)}{\tan\frac{\sigma-\sigma_1(s)}{2}}\, d\sigma,$$

the integrals taken in the principal-value sense.

Theorem

Let $a \in SL^{\alpha}$ *with* $\alpha \geq 2$ *and let* $\lambda_1^{(n)} \leq \ldots \leq \lambda_n^{(n)}$ *be the eigenvalues of* $T_n(a)$. *If* n *is sufficiently large, then*

(i) *the eigenvalues of* $T_n(a)$ *are all distinct, i.e.,* $\lambda_1^{(n)} < \lambda_2^{(n)} < \ldots < \lambda_n^{(n)}$,

(ii) *the numbers* $s_j^{(n)} := \psi(\lambda_j^{(n)})$ $(j = 1, \ldots, n)$ *satisfy*

$$(n+1)s_j^{(n)} + \eta(s_j^{(n)}) = \pi j + \Delta_1^{(n)}(j)$$

with $\Delta_1^{(n)}(j) = o(1/n^{\alpha-2})$ *as* $n \to \infty$, *uniformly with respect to* j,

(iii) *this equation has exactly one solution* $s_j^{(n)} \in [0, \pi]$ *for each* $j = 1, \ldots, n$.

To write down the individual asymptotics of the eigenvalues, we introduce the parameter

$$d := \frac{\pi j}{n+1}.$$

Note that the dependence of d on j and n is suppressed.

Theorem

Let $a \in SL^{\alpha}$ with $\alpha \geq 2$ and let $s_j^{(n)}$ be as in previous Theorem. Then

$$s_j^{(n)} = d + \sum_{k=1}^{[\alpha]-1} \frac{p_k(d)}{(n+1)^k} + \Delta_2^{(n)}(j)$$

where $\Delta_2^{(n)}(j) = o\left(1/n^{\alpha-1}\right)$ as $n \to \infty$ uniformly in j, $p_1(d) = -\eta(d), p_2 = \eta(d)\eta'(d)$. $[\alpha]$ is integer part of α.

That is for $2 \leq \alpha < 3$ we have

$$s_j^{(n)} = d - \frac{\eta(a)}{n+1} + o(1/n^{\alpha-1}), \quad d = \frac{\pi j}{n+1}.$$

For $3 \leq \alpha < 4$ we have

$$s_j^{(n)} = d - \frac{\eta(a)}{n+1} + \frac{\eta(d)\eta'(d)}{(n+1)^2} + o(1/n^{\alpha-1}).$$

e.t.c.

Theorem

Let $\alpha \geq 2$ *and* $a \in SL^{\alpha}$. *Then*

$$\lambda_j^{(n)} = \psi(d) + \sum_{k=1}^{[\alpha]-1} \frac{c_k(d)}{(n+1)^k} + \Delta_3^{(n)}(j) \qquad (1)$$

where $\Delta_3^{(n)}(j) = o\left(d(\pi - d)/n^{\alpha-1}\right)$ *as* $n \to \infty$, *uniformly in* $j = 1, 2, \ldots, n$, *and*

$$c_1(d) = -\psi'(d)\eta(d),$$
$$c_2(d) = \psi''(d)\eta^2(d)/2 + \psi'(d)\eta(d)\eta'(d).$$

Here is the result for the extreme eigenvalues.

Corollary

Let $a \in SL^{\alpha}$ with some $\alpha \geq 3$.
(i) *If $j/(n+1) \to 0$ then*

$$\lambda_j^{(n)} = \frac{c_5 j^2}{(n+1)^2} + \frac{c_6 j^2}{(n+1)^3} + \Delta_5^{(n)}(j),$$

where $c_5 = \pi^2 g''(0)/2$, $c_6 = -\pi^2 g''(0)\eta'(0)$, and $\Delta_5^{(n)}(j) = o(j/n^3)$ as $n \to \infty$.
(ii) *If $j/(n+1) \to 1$ then*

$$\lambda_j^{(n)} = M + \frac{c_7 (n+1-j)^2}{(n+1)^2} + \frac{c_8 (n+1-j)^2}{(n+1)^3} + \Delta_6^{(n)}(j),$$

where $c_7 = \pi^2 g''(\varphi_0)/2$, $c_8 = -\pi^2 g''(\varphi_0)\eta'(\pi)$, and $\Delta_6^{(n)}(j) = o(n+1-j/n^3)$ as $n \to \infty$.

This theorem is close to a result by Widom 1958, who considered the case where g is an even function and j is fixed.

Local nature of the asymptotics

Symmetric symbol: $\psi(\varphi) = g(\varphi) = (a(e^{i\varphi}))$

1. Normal case: $g'(\varphi) \neq 0$ $(\varepsilon < \frac{\pi j}{n+1} < \pi - \varepsilon)$. Inner eigenvalues

$$\lambda_j^{(n)} = g\left(\frac{\pi j}{n+1}\right) + \frac{c_1\left(\frac{\pi j}{n+1}\right)}{n+1} + O\left(n^{-2}\right).$$

Distance between next eigenvalues is

$$O\left(\frac{1}{n}\right)$$

2. Exceptional case: $\left(\frac{\pi j}{n+1}\right) \leq \varepsilon$. Extreme eigenvalue

$$\lambda_j^{(n)} = g\left(\frac{\pi j}{n+1}\right) + O\left(\frac{1}{n^3}\right), \quad \frac{\pi j}{n+1} \leq \varepsilon.$$

Distance between next eigenvalues is

$$O\left(\frac{1}{n^2}\right)$$

Asymptotics: is define by behavior of function $g(\varphi)$ in neighborhood of point φ_0, where eigenvalues are located.

More general restriction

1. Normal case:

$$g(\varphi) - g(\varphi_0) = (\varphi - \varphi_0)\,\tilde{g}(\varphi), \quad \tilde{g}(\varphi) \in W(= W^0).$$

   That is $g(\varphi) \in W^1$

2. Exceptional case:

$$g(\varphi) - g(\varphi_0) = (\varphi - \varphi_0)^2\,\tilde{g}(\varphi), \quad \tilde{g}(\varphi) \in W(= W^0).$$

Main ideas of the Proof

Lemma

Let $a \in SL^{\alpha}$, $\alpha \geq 2$ *and* $n \geq 1$. *A number* $\lambda = \psi(s)$ *is an eigenvalue of* $T_n(a)$ *if and only if*

$$e^{i(n+1)\sigma_2(s)}\Theta_{n+2}(e^{i\sigma_1(s)}, s)\widehat{\Theta}_{n+2}(e^{i\sigma_2(s)}, s)$$
$$-e^{i(n+1)\sigma_1(s)}\Theta_{n+2}(e^{i\sigma_2(s)}, s)\widehat{\Theta}_{n+2}(e^{i\sigma_1(s)}, s) = 0,$$

where, for every $k \geq 1$, *the functions* Θ_k *and* $\widehat{\Theta}_k$ *are defined by*

$$\Theta_k(t,s) := [T_k^{-1}(b(\cdot, s))\chi_0](t), \qquad \widehat{\Theta}_k(t,s) := [T_k^{-1}(\tilde{b}(\cdot, s))\chi_0](t^{-1}),$$

and $\tilde{b}(t,s) := b(1/t, s)$, $\chi_\ell(t) = t^{\ell}$, $\chi_0(t) = 1$.

Proof. We are searching for all values of λ belonging to $[0, M]$ such that the equation $T_n(a)X = \lambda X$ has non-zero solutions X in $L_2^{(n)}$. Using the change of variable $\lambda = \psi(s)$ we can rewrite the latter equation as

$$T_n(a - \psi(s))X = 0. \tag{2}$$

Equation (2) is equivalent to

$$P_n b(\cdot, s) p(\cdot, s) X = 0, \tag{3}$$

where $p(t, s) := e^{-is}(t - e^{i\sigma_1(s)})(t^{-1} - e^{-i\sigma_2(s)})$. Multiply (3) by the function χ_1 to get

$$(P_{n+1} - P_1) b(\cdot, s) \chi_1 p(\cdot, s) X = 0. \tag{4}$$

Here $P_{n+1} - P_1$ is just one way to write the orthogonal projection of the space $L_2(\mathbb{T})$ onto the span of $\chi_1, \ldots, \chi_n$. Note that $\chi_1 p(\cdot, s)X \in L_2^{(n+2)}$ and put

$$Y := T_{n+2}(a - \psi(s))\chi_1 X = P_{n+2} b(\cdot, s)\chi_1 p(\cdot, s)X = T_{n+2}(b(\cdot, s))\chi_1 p(\cdot, s)X.$$

Then (4) can be written as $(P_{n+1} - P_1)Y = 0$. This means that Y has the form

$$Y = y_0 \chi_0 + y_{n+1} \chi_{n+1}.$$

Since $T_{n+2}(b(\cdot, s))$ is invertible, it follows that $T_{n+2}^{-1}(b(\cdot, s))Y = \chi_1 p(\cdot, s)X$, that is,

$$y_0 [T_{n+2}^{-1}(b(\cdot, s))\chi_0](t) + y_{n+1}[T_{n+2}^{-1}(b(\cdot, s))\chi_{n+1}](t) = tp(t, s)X(t). \quad (5)$$

Now recall notation (5). Taking into account the identity

$$W_{n+2} T_{n+2}(b) W_{n+2} = T_{n+2}(\tilde{b}),$$

it is easy to verify that

$$[T_{n+2}^{-1}(b(\cdot, s))\chi_{n+1}](t) = t^{n+1}\widehat{\Theta}_{n+2}(t, s).$$

Therefore (5) can be written as

$$y_0 \Theta_{n+2}(t, s) + y_{n+1} t^{n+1} \widehat{\Theta}_{n+2}(t, s) = tp(t, s)X(t). \tag{6}$$

Thanks to the factor $p(t,s)$, the right-hand side vanishes at both $t = e^{i\sigma_1(s)}$ and $t = e^{i\sigma_2(s)}$. Consequently, y_0 and y_{n+1} must satisfy the homogeneous system of linear algebraic equations given by

$$\begin{aligned} \Theta_{n+2}(e^{i\sigma_1(s)}, s)y_0 + e^{i(n+1)\sigma_1(s)}\widehat{\Theta}_{n+2}(e^{i\sigma_1(s)}, s)y_{n+1} = 0, \\ \Theta_{n+2}(e^{i\sigma_2(s)}, s)y_0 + e^{i(n+1)\sigma_2(s)}\widehat{\Theta}_{n+2}(e^{i\sigma_2(s)}, s)y_{n+1} = 0. \end{aligned} \tag{7}$$

If $y_0 = y_{n+1} = 0$, then, by (6), the function X is zero. Therefore the initial equation (2) has a non-trivial solution X if and only if the determinant of system (7) is zero. □

Recall that $b_{\pm}(\cdot, s)$ are the Wiener-Hopf factors of $b(\cdot, s)$:

$$b(t,s) = b_+(t,s)\, b_-(t,s)$$

$$b_+(t,s) = \sum_{j=0}^{\infty} u_j(s) t^j \quad \text{and} \quad b_-(t,s) = \sum_{j=0}^{\infty} v_j(s) t^{-j}$$

$$T^{-1}(b(\cdot,s)) = b_+^{-1}(\cdot,s) P b_-^{-1}(\cdot,s),$$

$$[T^{-1}(b(\cdot,s))\chi_0](t) = [b_+^{-1}(\cdot,s) P b_-^{-1}(\cdot,s) X_0](t) = b_+^{-1}(t,s).$$

$$T_n(a)\ \bar{x}_n = \bar{f}_n$$

$$L_n^* \left(\operatorname{diag} \left(\lambda_j^{(n)} \right)_{j=1}^{n} \right) L_n \bar{X}_n = \bar{f}_n, \quad L^* = L_n^{-1}$$

$$\begin{pmatrix} \lambda_1^{(n)} & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^{(n)} & 0 & \dots & 0 \\ 0 & & \lambda_3^{(n)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & \lambda_n^{(n)} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{pmatrix}$$

$$\bar{Y} = L_n \bar{X}, \quad \bar{F} = L_n \bar{f}.$$

Singular Value Decomposition.

Example 1.

Consider the non-rational symbol

$$a(e^{i\sigma}) = g(\sigma) = g_2\sigma^2 + g_3\sigma^3 + g_4\sigma^{4+\beta} + g_5\sigma^5 + g_6\sigma^6 + g_7\sigma^7, \quad \sigma \in [0, 2\pi],$$

where $\beta \in [0, 1)$ and the coefficients $g_2, \ldots, g_7$ are chosen in such a manner that

$$g(2\pi) = g'(2\pi) = 0 \qquad \text{and} \qquad g^{(k)}(2\pi) = g^{(k)}(0) \quad \text{for} \quad k = 2, 3, 4.$$

Elementary computations yield

$$\begin{aligned}
g_2 &= (24 - 38\beta + 13\beta^2 + 2\beta^3 - \beta^4)/(2\pi)^2,\\
g_3 &= (24 - 50\beta + 35\beta^2 - 10\beta^3 + \beta^4)/(2\pi)^3,\\
g_4 &= 240/(2\pi)^{4+\beta},\\
g_5 &= (360 + 42\beta - 201\beta^2 + 42\beta^3 + 3\beta^4)/(2\pi)^5,\\
g_6 &= (-216 + 66\beta + 209\beta^2 - 54\beta^3 - 5\beta^4)/(2\pi)^6,\\
g_7 &= (48 - 20\beta - 50\beta^2 + 20\beta^3 + 2\beta^4)/(2\pi)^7.
\end{aligned}$$

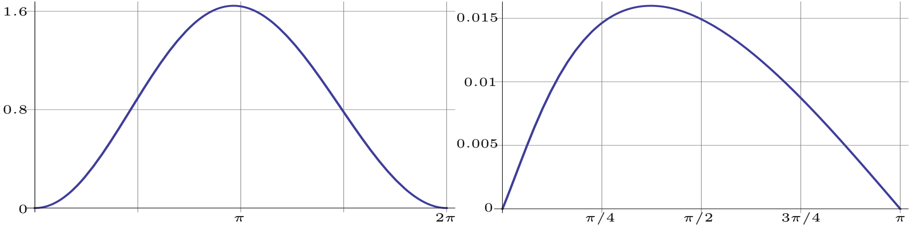

Figure: Graph of $g(\sigma) = a(e^{i\sigma})$ (left), and $\eta(s)$ (right) for $\beta = 1/5$.

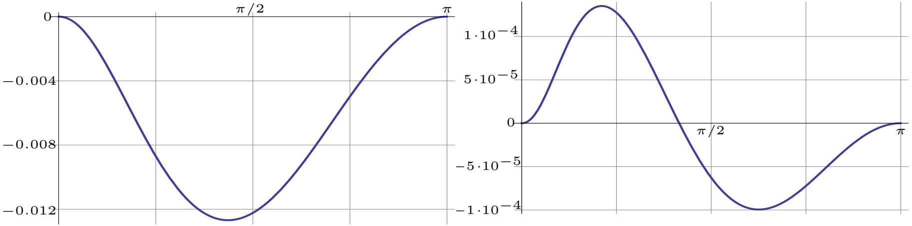

Figure: The functions $c_1(d)$ (left) and $c_2(d)$ (right).

n	64	128	512	1024	2048	4096
$\varepsilon^{(n,1)}$	$2.0\cdot10^{-4}$	$9.8\cdot10^{-5}$	$2.5\cdot10^{-5}$	$1.2\cdot10^{-5}$	$6.2\cdot10^{-6}$	$3.1\cdot10^{-6}$
$(n+1)\varepsilon^{(n,1)}$	$1.3\cdot10^{-2}$	$1.3\cdot10^{-2}$	$1.3\cdot10^{-2}$	$1.3\cdot10^{-2}$	$1.3\cdot10^{-2}$	$1.3\cdot10^{-2}$
$\varepsilon^{(n,2)}$	$3.2\cdot10^{-8}$	$8.1\cdot10^{-9}$	$5.1\cdot10^{-10}$	$1.3\cdot10^{-10}$	$3.2\cdot10^{-11}$	$8.1\cdot10^{-12}$
$(n+1)^2\varepsilon^{(n,2)}$	$1.4\cdot10^{-4}$	$1.4\cdot10^{-4}$	$1.4\cdot10^{-4}$	$1.4\cdot10^{-4}$	$1.4\cdot10^{-4}$	$1.4\cdot10^{-4}$
$\varepsilon^{(n,3)}$	$2.3\cdot10^{-10}$	$1.3\cdot10^{-11}$	$4.1\cdot10^{-14}$	$2.2\cdot10^{-15}$	$2.4\cdot10^{-16}$	$3.0\cdot10^{-17}$
$(n+1)^3\varepsilon^{(n,3)}$	$6.2\cdot10^{-5}$	$2.8\cdot10^{-5}$	$5.5\cdot10^{-6}$	$2.4\cdot10^{-6}$	$2.1\cdot10^{-6}$	$2.1\cdot10^{-6}$
$\hat{\varepsilon}^{(n)}$	$2.3\cdot10^{-10}$	$1.3\cdot10^{-11}$	$4.1\cdot10^{-14}$	$2.2\cdot10^{-15}$	$1.2\cdot10^{-16}$	$6.7\cdot10^{-18}$
$(n+1)^{4.2}\hat{\varepsilon}^{(n)}$	$9.3\cdot10^{-3}$	$9.6\cdot10^{-3}$	$9.8\cdot10^{-3}$	$9.8\cdot10^{-3}$	$9.9\cdot10^{-3}$	$9.9\cdot10^{-3}$

Table: Maximum errors and normalized maximum errors for the eigenvalues of $T_n(a)$ obtained with our formula (1), $\varepsilon^{(n,p)}$ with $p = 1, 2, 3$, and by fixed-point iterations, $\hat{\varepsilon}^{(n)}$, for different values of n. The data were obtained by comparison with the solutions given by *Wolfram Mathematica*.

Note that Table 1 shows that $\hat{\varepsilon}^{(n)} = O(1/(n+1)^{4.2})$ as $n \to \infty$.

Eigenvectors

For $\gamma \in \mathbb{R}$, we define z_k^γ as $e^{i\sigma_k(s_j^{(n)})\gamma}$. Given a function $f : \mathbb{T} \to \mathbb{C}$, let f_p be its pth Fourier coefficient, and for a vector X, let X_p stand for its pth component. Let $\theta = (\theta_p)_{p=0}^{n+1}$ be the vector in the first column of the matrix $T_{n+2}^{-1}(b(\cdot, s_j^{(n)}))$. For $t \in \mathbb{C}$, we put

$$\theta(t) = \theta_0 + \theta_1 t + \cdots + \theta_{n+1} t^{n+1}.$$

The following theorem describes the components of the eigenvectors of $T_n(a)$.

Theorem

Let $a \in SL^\alpha$. The vector

$$X^{(n,j)} = M^{(n,j)} + L^{(n,j)} + R^{(n,j)} \tag{8}$$

whose p-th component, $p = 0, 1, \ldots, n-1$, is given by

$$M_p^{(n,j)} := z_1^{\frac{n-1}{2}-p}|\theta(z_1)| + (-1)^{n-j} z_2^{\frac{n-1}{2}-p}|\theta(z_2)|,$$

$$L_p^{(n,j)} := -\frac{z_1^{\frac{n+1}{2}}\overline{\theta(z_1)}}{2\pi i|\theta(z_1)|}\int_{\mathbb{T}}\left(\frac{\theta(t)-\theta(z_1)}{t-z_1} - \frac{\theta(t)-\theta(z_2)}{t-z_2}\right)\frac{dt}{t^{p+1}},$$

$$R_p^{(n,j)} := \overline{L_{n-p-1}^{(n,j)}},$$

is an eigenvector of $T_n(a)$ corresponding to the eigenvalue $\lambda_j^{(n)}$. Moreover, $M^{(n,j)}$ is conjugate symmetric, i.e., $M_p^{(n,j)} = \overline{M_{n-p-1}^{(n,j)}}$.

Theorem

Let $a \in SL^{\alpha}$. *For* $k = 1, 2$, *let* $\hat{z}_k := e^{i\sigma_k(\hat{s}_j^{(n)})}$, *and, for* $p = 0, 1, \ldots, n-1$, *put*

$$\hat{M}_p^{(n,j)} := \frac{\hat{z}_1^{\frac{n-1}{2}-p}}{|b_+(\hat{z}_1)|} + (-1)^{n-j}\frac{\hat{z}_2^{\frac{n-1}{2}-p}}{|b_+(\hat{z}_2)|},$$

$$\hat{L}_p^{(n,j)} := -\frac{\hat{z}_1^{\frac{n+1}{2}} b_+(\hat{z}_1)}{2\pi i|b_+(\hat{z}_1)|}\int_{\mathbb{T}}\left(\frac{b_+^{-1}(t) - b_+^{-1}(\hat{z}_1)}{t - \hat{z}_1} - \frac{b_+^{-1}(t) - b_+^{-1}(\hat{z}_2)}{t - \hat{z}_2}\right)\frac{dt}{t^{p+1}},$$

$$\hat{R}_p^{(n,j)} := \overline{\hat{L}_{n-1-p}^{(n,j)}}.$$

Then there is a vector $\Omega_1^{(n,j)}$ *such that* $[\Omega_1^{(n,j)}]_p = o(1/n^{\alpha-3})$ *as* $n \to \infty$, *uniformly in* j *and* p, *and such that*

$$X^{(n,j)} = \hat{M}^{(n,j)} + \hat{L}^{(n,j)} + \hat{R}^{(n,j)} + \Omega_1^{(n,j)} \tag{9}$$

is an eigenvector of $T_n(a)$ *corresponding to the eigenvalue* $\lambda_j^{(n)}$.

Symbols with Fisher–Harturg singularity.

$$a_{\alpha,\beta}(t) = (1-t)^{\alpha}(-t)^{\gamma}, \qquad 0 < \alpha < |\beta| < 1.$$

Conjecture of
H.Dai, Z.Geary and L.P.Kadanoff, 2009

$$\lambda_j^{(n)} \sim a_{\alpha,\beta}\left(\omega_j \cdot \exp\left\{(2\alpha+1)\,\frac{\log}{n}\right\}\right),$$

where $\omega_j = \exp\left(-i\,\frac{2\pi j}{n}\right)$.

Complex value case

$$a(t) = t^{-1}(1-t)^{\alpha} f(t), \quad \alpha \in R_{+} \backslash N$$

where

1. $f(t) \in H^{\infty} \cap C^{\infty}$.
2. f can be analytically extended to a neighborhood of $\mathbb{T} \backslash \{1\}$.
3. The range of the symbol a $\mathcal{R}(a)$ is a closed Jordan curve without loops and winding number -1 around each interior point.

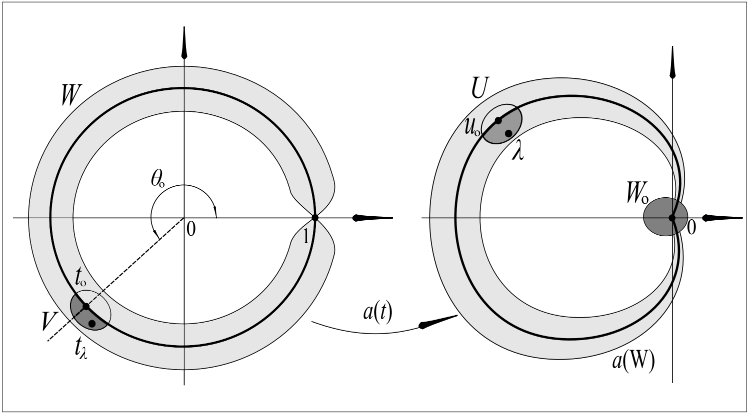

Figure: The map $a(t)$ over the unit circle.

Lemma

Let $a(t) = t^{-1}h(t)$ *be a symbol that satisfies the following conditions:*

1. $h \in H^\infty$.
2. $\mathcal{R}(a)$ *is a closed Jordan curve in* $\mathbb{C}$ *without loops.*
3. $wind_\lambda(a) = -1$, *for each* λ *in the interior of sp* $T(a)$.

Then, for each λ *in the interior of sp* $T(a)$, *we have the equality*

$$D_n(a - \lambda) = (-1)^n h_o^{n+1} \left[\frac{1}{h(t) - \lambda t} \right]_n,$$

for every $n \in \mathbb{N}$.

Theorem

We have the following asymptotic expression for λ_j*:*

$$\lambda_j = a(\omega_j) + (\alpha+1)\omega_j a'(\omega_j)\frac{\log(n)}{n} + \frac{\omega_j a'(\omega_j)}{n}\log\left(\frac{a^2(\omega_j)}{c_o a'(\omega_j)\omega_j^2}\right) + \mathcal{O}\left(\frac{\log(n)}{n}\right)^2, \ n\to\infty,$$

where $\omega_j = \exp\left(-i\,\frac{2\pi j}{n}\right)$.

$n = 4096$

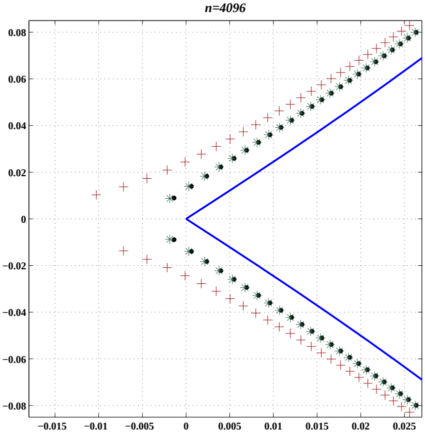

Figure: The solid blue line is the range of a. The black dots are sp $T_n(a)$ calculated by *Matlab*. The red crosses and the green stars are the approximations, for 1 and 2 terms respectively. Here we took $\alpha = 3/4$.

New problems

1. Symbols of the kind:

$$a(t) = \frac{a_{-1}}{t} + a_0 + a_1 t + a_2 t^2$$

$$a_0(t) = \frac{1}{t} + t^2$$

2. Symbols of the kind:

$$a(t) = \frac{a_{-2}}{t^2} + \frac{a_{-1}}{t} + a_0 + a_1 t + a_2 t^2$$

$$a_0(t) = \left(\frac{1}{t} - 2 + t\right)^2$$

3. No simpleloop case.

4. Fisher-Harturg general case

$$a(t) = (t - t_0)^{\alpha} t^{\beta}, \quad \alpha, \beta \in R \; (\alpha, \beta \in \mathbb{C}).$$