On the Toeplitz operators with piecewise continuous symbols on the Bergman space

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To I. B. Simonenko in occasion of his 70th birthday.

Abstract. The paper is devoted to the study of Toeplitz operators with piecewise continuous symbols. We clarify the geometric regularities of the behaviour of the essential spectrum of Toeplitz operators in dependence on their crucial data: the angles between jump curves of symbols at a boundary point of discontinuity and on the limit values reached by a symbol at that boundary point. We show then that the curves supporting the symbol discontinuities, as well as the number of such curves meeting at a boundary point of discontinuity, do not play any essential role for the Toeplitz operator algebra studied. Thus we exclude the curves of symbol discontinuity from the symbol class definition leaving only the set of boundary points (where symbols may have discontinuity) and the type of the expected discontinuity. Finally we describe the C^*-algebra generated by Toeplitz operators with such symbols.

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1. Introduction

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and $\gamma = \partial\mathbb{D}$ be its boundary. Consider $L_2(\mathbb{D})$ with respect to the standard Lebesgue plane measure and its subspace, the Bergman space $\mathcal{A}^2(\mathbb{D})$, which consists of functions analytic in $\mathbb{D}$. Let $B_{\mathbb{D}}$ stand for the orthogonal Bergman projection of $L_2(\mathbb{D})$ onto $\mathcal{A}^2(\mathbb{D})$. Given a function $a(z) \in L_\infty(\mathbb{D})$, the Toeplitz operator T_a with symbol $a = a(z)$ is defined on $\mathcal{A}^2(\mathbb{D})$ as follows:

$$T_a : \varphi \in \mathcal{A}^2(\mathbb{D}) \longmapsto B_{\mathbb{D}}(a\varphi) \in \mathcal{A}^2(\mathbb{D}).$$

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In the paper we study Toeplitz operators with piecewise continuous symbols. The first results in this direction date from the early 1980s (see [3, 4, 5, 6]) and show that essentially the situation is the same as in the case of Toeplitz operators with piecewise continuous symbols on the Hardy space. The exact result is given in Theorem 2.2 below.

The next essential advance in this direction was made by M. Loaiza [2] after about 20 years of silence. She described the case of piecewise continuous symbols having more then two limit values at the boundary point of discontinuity. This result was made possible due to recent work [7] describing the commutative C^*-algebras of Toeplitz operators on the Bergman space.

We recall that for piecewise continuous symbols the product of two Toeplitz operators is not in general a compact perturbation of a Toeplitz operator. Thus the algebra generated by such operators has a quite complicated structure, coinciding with the uniform closure of the set of all elements of the form

$$\sum_{k=1}^{p} \prod_{j=1}^{q_k} T_{a_{j,k}}. \tag{1.1}$$

It is very interesting and important to understand the nature of the operators forming the algebra and, in particular, to know *whether this Toeplitz operator algebra contains any other Toeplitz operator, apart from its initial generators.* Note that this question has remained unanswered since the very first work on the subject.

In the paper we present some recent advances in the area. In Section 2 we recall the previous results, especially on algebras generated by Toeplitz operators with piecewise continuous symbols, which are relevant to the main content of the paper. In Section 3 we show how the results of [2] allow us to understand the geometric regularities of the behaviour of the essential spectrum of Toeplitz operators in dependence on their crucial data: the angles between jump curves of symbols at a boundary point of discontinuity and on the limit values reached by a symbol at that boundary point. Section 4 is devoted to the local analysis of Toeplitz operators at a point of discontinuity. The results of [7] permit us to get a highly unexpected result, which partially answers the above question. We show that the closure of elements of the form (1.1) contains many Toeplitz operators, and the symbols of these Toeplitz operators belong to a much wider class of discontinuous functions, as compared with the symbols of the initial generators. In particular, it turns out that the algebra considered in [6] already contained all operators from the algebra considered in [2], though previously there were no means to realize this fact. The main conclusion of the section is that we can start from very different sets of symbols and obtain exactly the same operator algebra as a result. That is, the curves supporting the symbol discontinuities, as well as the number of such curvess meeting at a boundary point of discontinuity, do not play in fact any essential role for the Toeplitz operator algebra studied. This observation motivates us to exclude the curves of symbol discontinuity from the very beginning and to leave in the symbol class definition only the set of boundary points (where symbols may have discontinuity) and the type of the expected discontinuity. We do this in the final Section 5 introducing the so-called boundary piecewise continuous symbols and describing the algebra generated by Toeplitz operators with such symbols.

2. Preliminaries

In this section we recall some well known results relevant to the main content of the paper.

Given a linear space (or algebra) $\mathcal{A} \subset L_\infty(\mathbb{D})$, we denote by $\mathcal{T}(\mathcal{A})$ the C^*-algebra generated by all Toeplitz operators T_a with $a \in \mathcal{A}$, and we denote by $\mathrm{Sym}\, \mathcal{T}(\mathcal{A}) = \mathcal{T}(\mathcal{A})/\mathcal{K}$ its (Fredholm) symbol, or Calkin algebra. Here $\mathcal{K}$ is the ideal of all compact operators on $\mathcal{A}^2(\mathbb{D})$.

We start with the description of the algebra generated by Toeplitz operators with continuous symbols, which goes back to L. Coburn [1].

Theorem 2.1. *The algebra $\mathcal{T}_C = \mathcal{T}(C(\overline{\mathbb{D}}))$ is irreducible and contains the entire ideal $\mathcal{K}$ of compact operators on $\mathcal{A}^2(\mathbb{D})$. Each operator $T \in \mathcal{T}(C(\overline{\mathbb{D}}))$ is of the form*

$$T = T_a + K,$$

where $a \in C(\overline{\mathbb{D}})$ and K is a compact operator. The homomorphism

$$\mathrm{sym} : \mathcal{T}_C \longrightarrow \mathrm{Sym}\, \mathcal{T}_C = \mathcal{T}_C/\mathcal{K} \cong C(\gamma)$$

is given by

$$\mathrm{sym} : T = T_a + K \longmapsto a_{|_\gamma}.$$

The operator $T \in \mathcal{T}_C$ is Fredholm if and only if its symbol is invertible, i.e., the function $\mathrm{sym}\, T \neq 0$ on γ, and

$$\mathrm{Ind}\, T = -\frac{1}{2\pi}\{\mathrm{sym}\, T\}_\gamma.$$

The situation changes if we extend the symbol class from continuous to piecewise continuous functions. The corresponding results were obtained in [3, 4, 5, 6]. To introduce them we proceed as follows. Denote by ℓ a union of a finite number of piecewise smooth curves in $\overline{\mathbb{D}}$. We will assume that the intersection $\gamma \cap \ell$ consists of a finite number of endpoints of ℓ: $T = \gamma \cap \ell = \{t_1, ..., t_m\}$, and each $t_p \in T$ is the endpoint for only one curve from ℓ.

Denote by $PC(\overline{\mathbb{D}}, \ell)$ the algebra of all functions $a(z)$, continuous in $\overline{\mathbb{D}} \setminus \ell$, and having left and right limit values at all points of ℓ. In particular, at each point $t_p \in T$ any function $a \in PC(\overline{\mathbb{D}}, \ell)$ has two limit values: $a(t_p - 0)$ and $a(t_p + 0)$, following the positive orientation of γ.

Let $\widehat{\gamma}$ be the boundary γ, cut by points $t_p \in T$. The pair of points, which correspond to a point $t_p \in T$, $p = \overline{1, m}$, we denote by $t_p - 0$ and $t_p + 0$, following the positive orientation of γ. Let $\overline{X} = \sqcup_{p=1}^m \Delta_p$ be the disjoint union of segments $\Delta_p = [0, 1]$. Denote by Γ the union $\widehat{\gamma} \cup \overline{X}$ with the following point identification

$$t_p - 0 \equiv 0_p, \qquad t_p + 0 \equiv 1_p,$$

where $t_p \pm 0 \in \widehat{\gamma}$, 0_p and 1_p are the endpoints of Δ_p, $p = 1, 2, ..., m$.

Theorem 2.2. *The C^*-algebra $\mathcal{T}_{PC} = \mathcal{T}(PC(\overline{\mathbb{D}}, \ell))$ is irreducible and contains the ideal $\mathcal{K}$ of compact operators. The (Fredholm) symbol algebra* $\operatorname{Sym}\mathcal{T}_{PC} = \mathcal{T}_{PC}/\mathcal{K}$ *is isomorphic to the algebra $C(\Gamma)$. The homomorphism*

$$\text{sym} : \mathcal{T}_{PC} \to \operatorname{Sym}\mathcal{T}_{PC} = \mathcal{T}_{PC}/\mathcal{K} \cong C(\Gamma)$$

is generated by the following mapping of generators of $\mathcal{T}_{PC}$

$$\text{sym} : T_a \longmapsto \begin{cases} a(t), & t \in \widehat{\gamma} \\ a(t_p - 0)(1 - x) + a(t_p + 0)x, & x \in [0, 1] \end{cases},$$

where $t_p \in \ell \cap \gamma$, $p = 1, 2, ..., m$.

An operator $T \in \mathcal{T}_{PC}$ is Fredholm if and only if its symbol is invertible, i.e., the function $\operatorname{sym} T \neq 0$ *on Γ, and*

$$\operatorname{Ind} T = -\frac{1}{2\pi}\{\operatorname{sym} T\}_{\Gamma}.$$

We note that for piecewise continuous symbols the product of two Toeplitz operators is in general not longer a compact perturbation of a Toeplitz operator. The algebra $\mathcal{T}_{PC}$ does not coincide with the set of all operators of the form $T_a + K$ as in case of continuous symbols. It has a much more complicated structure, coinciding with the uniform closure of the set of all elements of the form

$$\sum_{k=1}^{p} \prod_{j=1}^{q_k} T_{a_{j,k}}, \tag{2.1}$$

where $a_{j,k} \in PC(\overline{D}, \ell)$, $p, q_k \in \mathbb{N}$.

At this stage an important question arises: *does the algebra $\mathcal{T}_{PC}$ contain any other Toeplitz operator, apart from its initial generators?*

An unexpected (partial) answer to this question will be provided in last two sections of the paper.

The key result permitting one to handle local situations for a wider class of discontinuous symbols was given in [7] and is as follows.

We start from $L_2(\Pi)$ over the upper half-plane Π with the usual Lebesgue plane measure and its Bergman subspace $\mathcal{A}^2(\Pi)$. Denote by $\mathcal{A}_\infty$ the C^*-algebra of bounded measurable homogeneous functions on Π of order zero, or functions depending only on the polar coordinate θ. Introduce the Toeplitz operator algebra $\mathcal{T}(\mathcal{A}_\infty)$, which is generated by all operators T_a with $a(\theta) \in \mathcal{A}_\infty$.

Theorem 2.3. *Let $a = a(\theta) \in \mathcal{A}_\infty$. Then the Toeplitz operator T_a acting on $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_a I = R\,T_a R^*$ acting on $L_2(\mathbb{R})$. The function $\gamma_a(\lambda)$ is given by*

$$\gamma_a(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_0^\pi a(\theta)\, e^{-2\lambda\theta}\, d\theta, \qquad \lambda \in \mathbb{R}. \tag{2.2}$$

Analyzing formula (2.2) we note that for each $a(\theta) \in L_\infty(0,\pi)$ the function $\gamma_a(\lambda)$ is continuous at all finite points $\lambda \in \mathbb{R}$. For a "very large λ" ($\lambda \to +\infty$) the exponent $e^{-2\lambda\theta}$ has a very sharp maximum at the point $\theta = 0$, and thus the major contribution to the integral in (2.2) for these "very large λ" is determined by values of $a(\theta)$ in a neighborhood of the point 0. The major contribution for a "very large negative λ" ($\lambda \to -\infty$) is determined by values of $a(\theta)$ in a neighborhood of π, due to a very sharp maximum of $e^{-2\lambda\theta}$ at $\theta = \pi$ for these values of λ. In particular, if $a(\theta)$ has limits at the points 0 and π, then

$$\begin{aligned} \lim_{\lambda\to+\infty} \gamma_a(\lambda) &= \lim_{\theta\to 0} a(\theta), \\ \lim_{\lambda\to-\infty} \gamma_a(\lambda) &= \lim_{\theta\to \pi} a(\theta). \end{aligned}$$

Corollary 2.4. *The algebra $\mathcal{T}(\mathcal{A}_\infty)$ is commutative. The isomorphic imbedding*

$$\tau_\infty : \mathcal{T}(\mathcal{A}_\infty) \longrightarrow C_b(\mathbb{R})$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(\mathcal{A}_\infty)$

$$\tau_\infty : T_a \longmapsto \gamma_a(\lambda),$$

where $a = a(\theta) \in \mathcal{A}_\infty$.

The above result was the starting point for the study of the algebra generated by Toeplitz operators with piecewise continuous symbols having more than two limit values at the boundary points, done by M. Loaiza [2]. We list here the principal local situation and the final result in a form convenient for us.

Via a Möbius transformation the principal local situation in [2] is reduced to the following upper half-plane setting. Given a finite number of different points on $[0,\pi]$,

$$0 = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_{n-1} < \theta_n = \pi,$$

we denote by $\mathcal{A}(\Lambda)$ with $\Lambda = \{\theta_1, \theta_2, \dots, \theta_{n-1}\}$ the algebra of piecewise constant functions on $[0,\pi]$ with jump points in Λ, and let $H(\mathcal{A}(\Lambda))$ be the algebra of homogeneous of zero order functions on Π whose restrictions onto the upper half-circle (parameterized by $\theta \in [0,\pi]$) belong to $\mathcal{A}(\Lambda)$. Note that each (piecewise constant) function $a \in H(\mathcal{A}(\Lambda))$ has n limit values at the origin.

Denote by V_k, $k = 1, 2, \dots, n$, the cone on the upper half-plane Π, supported on $(\theta_{k-1}, \theta_k]$. Then the n-dimensional algebra $H(\mathcal{A}(\Lambda))$ consists of all functions having the form

$$a(z) = a_1\chi_{V_1}(z) + a_2\chi_{V_2}(z) + \dots + a_n\chi_{V_n}(z),$$

where $(a_1, a_2, \dots, a_n) \in \mathbb{C}^n$, and $\chi_k(z)$ are the characteristic functions of the cones V_k, $k = 1, 2, \dots, n$.

The Toeplitz C^*-algebra $\mathcal{T}(H(\mathcal{A}(\Lambda)))$ is obviously generated by n *commuting* Toeplitz operators $T_{\chi_{V_k}}$, $k = 1, 2, \dots, n$, and we have

$$\gamma_{\chi_{V_k}}(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_{\theta_k - 1}^{\theta_k} e^{-2\lambda\theta}\, d\theta = \frac{e^{-2\theta_k\lambda} - e^{-2\theta_{k-1}\lambda}}{e^{-2\pi\lambda} - 1}, \qquad \lambda \in \mathbb{R}. \tag{2.3}$$

Each function $\gamma_{\chi_{V_k}}$ is continuous on $\overline{\mathbb{R}}$ and

$$\begin{array}{lll} \lim\limits_{\lambda\to-\infty}\gamma_{\chi_{V_1}}(\lambda)=0, & \lim\limits_{\lambda\to+\infty}\gamma_{\chi_{V_1}}(\lambda)=1, & \\ \lim\limits_{\lambda\to-\infty}\gamma_{\chi_{V_k}}(\lambda)=0, & \lim\limits_{\lambda\to+\infty}\gamma_{\chi_{V_k}}(\lambda)=0, & k=2,3....,n-1, \\ \lim\limits_{\lambda\to-\infty}\gamma_{\chi_{V_n}}(\lambda)=1, & \lim\limits_{\lambda\to+\infty}\gamma_{\chi_{V_n}}(\lambda)=0. & \end{array}$$

Furthermore, each function $\gamma_{\chi_{V_k}}$ is non-negative and

$$\sum_{k=0}^{n}\gamma_{\chi_{V_k}}(\lambda)\equiv 1;$$

thus the set

$$\Delta(\Lambda)=\{t=(t_1,t_2,...,t_n):t_k=\gamma_{\chi_{V_k}}(\lambda),\quad \lambda\in\overline{\mathbb{R}},\quad k=1,...,n\} \tag{2.4}$$

is a continuous curve lying on the standard $(n-1)$-dimensional simplex, and connecting the vertices $(1,0,...,0)$ and $(0,...,0,1)$.

In the following figure we present the behaviour of the set $\Delta(\Lambda)$ for the case n=3 in dependence of the angles (θ_1,θ_2).

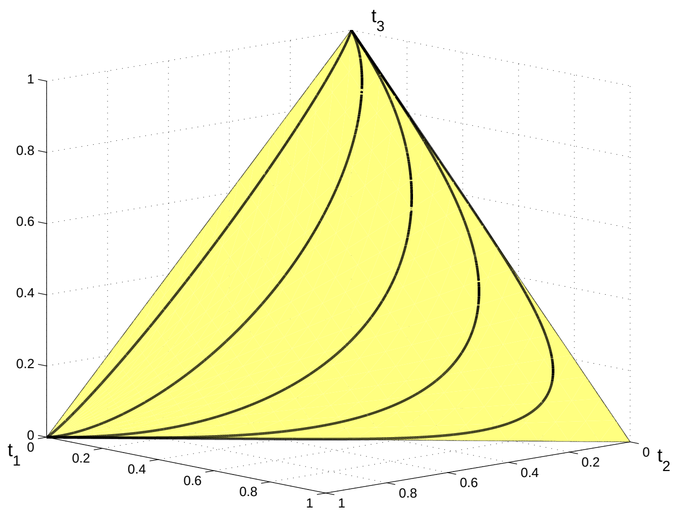

FIGURE 1. The angles (θ_1,θ_2) left to right: $(0.48\pi,0.52\pi)$, $(0.4\pi,0.6\pi)$, $(0.3\pi,0.7\pi)$, $(0.2\pi,0.8\pi)$, $(0.1\pi,0.9\pi)$.

Theorem 2.5. *Given a set $\Lambda=\{\theta_1,\theta_2,...,\theta_{n-1}\}$, the Toeplitz C^*-algebra $\mathcal{T}(H(\mathcal{A}(\Lambda)))$ is isomorphic and isometric to $C(\Delta(\Lambda))$. The isomorphism*

$$\tau:\mathcal{T}(H(\mathcal{A}(\Lambda)))\longrightarrow C(\Delta(\Lambda))$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(H(\mathcal{A}(\Lambda)))$: if $a(z) = a_1\chi_{V_1}(z) + a_2\chi_{V_2}(z) + ... + a_n\chi_{V_n}(z)$, then

$$\tau : T_a \longmapsto a_1t_1 + a_2t_2 + ... + a_nt_n,$$

where $t = (t_1, t_2, ..., t_n) \in \Delta(\Lambda)$.

Proof. The C^*-algebra $\mathcal{T}(H(\mathcal{A}(\Lambda)))$ is commutative, and is generated by n operators $T_{\chi_{V_k}}$, $k = 1, 2, ..., n$. Thus it is isomorphic and isometric to the algebra of all continuous functions on the joint spectrum of the above operators, which coincides obviously with $\Delta(\Lambda)$. □

Consider now the general case of Toeplitz operators with piecewise continuous symbols. Denote by ℓ a piecewise smooth curve in the closed unit disk $\overline{\mathbb{D}}$, satisfying the following properties: there are a finite number of points (nodes), which divide ℓ into simple oriented smooth curves ℓ_j, $j = \overline{1,k}$. We assume that the endpoints of ℓ are among the nodes. We will refer to a node using symbols u_{q,r_q}, where r_q is the number of curves meeting at this node, and q corresponds the node numbering. Denote by T the set of all nodes from $\ell \cap \gamma$, and assume that T consists of m points. For each node $t_{q,r_q-1} \in T$ there are $r_q - 1$ curves meeting at t_{q,r_q-1}, $q = 1, ..., m$. We assume as well that locally near t_{q,r_q-1} these curves are hypercycles, that is, there is a Möbius transformation of the unit disk to the upper half-plane under which the node t_{q,r_q-1} goes to the origin and the curves meeting at t_{q,r_q-1} are mapped to curves which near origin are straight line segments meeting at the origin.

Let now $PC(\overline{\mathbb{D}}, \ell)$ be the algebra of all functions $a(z)$, continuous in $\overline{\mathbb{D}} \setminus \ell$, and having left and right limit values at all points of ℓ_j: $a^+(z)$ and $a^-(z)$. On the nodes of type $t_{q,r_q-1} \in T$ the functions from $PC(\overline{\mathbb{D}}, \ell)$ have r limit values. We denote them by $a^{(1)}_{t_{q,r_q-1}}$, ..., $a^{(r)}_{t_{q,r_q-1}}$, counting counter-clockwise.

Let $\mathcal{T} = \mathcal{T}(PC(\overline{\mathbb{D}}, \ell))$ be the C^*-algebra generated by all Toeplitz operators T_a with symbols $a \in PC(\overline{\mathbb{D}}, \ell)$.

For each node $t_{q,r_q-1} \in T$ introduce the ordered set

$$\Lambda_q = \{\theta_1, \theta_2, ..., \theta_{r_q-1}\}$$

of the angles which the $r_q - 1$ curves meeting at the node t_{q,r_q-1} form with the boundary γ, counting them counter-clockwise. Introduce as well the corresponding curve

$$\Delta(\Lambda_q) = \{(t_1, t_2, ..., t_{r_q}) : t_k = \gamma_{\chi_{V_k}}(\lambda), \quad \lambda \in \overline{\mathbb{R}}, \quad k = 1, ..., r_q\}, \tag{2.5}$$

where each $t_k = \gamma_{\chi_{V_k}}(\lambda)$, $k = 1, ..., r_q$, is given by, see (2.3),

$$t_k = \gamma_{\chi_{V_k}}(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_{\theta_k - 1}^{\theta_k} e^{-2\lambda\theta}\, d\theta = \frac{e^{-2\theta_k\lambda} - e^{-2\theta_{k-1}\lambda}}{e^{-2\pi\lambda} - 1}, \qquad \lambda \in \mathbb{R},$$

Observe that the curve $\Delta(\Lambda_q)$ lies on the standard $(r_q - 1)$-dimensional simplex and connects its vertices $(1, 0, ..., 0)$ and $(0, ..., 0, 1)$.

Denote by $\widehat{\gamma}$ the set γ, cut by points $t_{q,r_q-1} \in T = \ell \cap \gamma$. The pair of points which correspond to a point $t_{q,r_q-1} \in T$ we denote by $t_{q,r_q-1} - 0$ and $t_{q,r_q-1} + 0$, following the positive orientation of γ. Let $\overline{X} = \cup_q \Delta(\Lambda_q)$ be the disjoint union of the sets (2.5). Denote by Γ the union $\widehat{\gamma} \cup \overline{X}$ with the following point identification

$$t_{q,r_q-1} - 0 \equiv (1, 0, ..., 0) \qquad t_{q,r_q-1} + 0 \equiv (0, ..., 0, 1),$$

where $t_{q,r_q-1} \pm 0 \in \widehat{\gamma}$, and $(1, 0, ..., 0)$ and $(0, ..., 0, 1)$ are the vertices of $\Delta(\Lambda_q)$.

Now the final result reads as follows.

Theorem 2.6. *The C^*-algebra $\mathcal{T} = \mathcal{T}(PC(\overline{\mathbb{D}}, \ell))$ is irreducible and contains the ideal $\mathcal{K}$ of compact operators. The symbol algebra* $\operatorname{Sym}\mathcal{T} = \mathcal{T}/\mathcal{K}$ *is isomorphic to the algebra $C(\Gamma)$. Identifying them, the symbol homomorphism*

$$\operatorname{sym} : \mathcal{T} \to \operatorname{Sym}\mathcal{T} = C(\Gamma)$$

is generated by the following mapping of generators of $\mathcal{T}$

$$\operatorname{sym} : T_a \longmapsto \begin{cases} a(t), & t \in \widehat{\gamma} \\ a^{(1)}_{t_{q,r_q-1}} t_1 + a^{(2)}_{t_{q,r_q-1}} t_2 + ... + a^{(r_q)}_{t_{q,r_q-1}} t_{r_q}, & t = (t_1, t_2, ..., t_{r_q}) \in \Delta(\Lambda_q) \end{cases},$$

where $t_{q,r_q-1} \in T$.

An operator $T \in \mathcal{T}$ is Fredholm if and only if its symbol is invertible, i.e., the function $\operatorname{sym} T \neq 0$ *on Γ, and*

$$\operatorname{Ind} T = -\frac{1}{2\pi}\{\operatorname{sym} T\}_{\Gamma}.$$

3. Essential and local spectra

The results given by Theorem 2.6 permit us, in particular, to describe easily the essential spectrum of T_a and to understand the geometric regularities of its behaviour.

Indeed, given a symbol $a \in PC(\overline{\mathbb{D}}, \ell)$, the essential spectrum $\operatorname{ess} - \operatorname{sp} T_a$ of the operator T_a, which is obviously equal to $\operatorname{Im} \operatorname{sym} T_a$, consists of two parts. Its regular part is the image of the symbol restricted on the boundary points of continuity, i.e., $\operatorname{sym} T_a|_{\widehat{\gamma}} = a|_{\widehat{\gamma}}$. The complementary part is a finite number of additional arcs, each one of which is the restriction of $\operatorname{sym} T_a$ onto the curve $\Delta(\Lambda_q)$, corresponding to the boundary point of discontinuity t_{q,r_q-1}.

We note that each such curve $\operatorname{sym} T_a|_{\Delta(\Lambda_q)}$ describes as well the spectrum of the local representative at the point t_{q,r_q-1} of the initial operator T_a.

Let us assume that t_0 is a boundary point of discontinuity for functions from $PC(\overline{\mathbb{D}}, \ell)$ an which n curves from ℓ intersect. As previously, introduce the ordered set

$$\Lambda = \{\theta_1, \theta_2, ..., \theta_{n-1}\}$$

of the angles which the above n curves form with the boundary γ, counting them counter-clockwise. As above, we add $\theta_0 = 0$ and $\theta_n = \pi$. Given a symbol $a \in$

$PC(\overline{\mathbb{D}}, \ell)$, introduce the ordered set

$$A = \{a_1, a_2, ..., a_n\},$$

where each a_k, $k = 1, 2, ..., n$, is the limit value of a at the point t_0 reached from the region between the $(k-1)$-th and k-th curves.

The local representative at the point t_0 of the operator T_a can be taken as the Toeplitz operator $T_{A,\Lambda}$ with piecewise constant symbol

$$a_{A,\Lambda}(\theta) = a_1\chi_{V_1}(\theta) + ... + a_n\chi_{V_n}(\theta) \in H(\mathcal{A}(\Lambda)),$$

where each χ_{V_k} is the characteristic function of the cone V_k supported on $(\theta_{k-1}, \theta_k]$.

That is the spectrum of $T_{a_{A,\Lambda}}$, which the same as the corresponding portion of the essential spectrum of T_a, is governed by the sets A and Λ and is given by the formula

$$\operatorname{sp} T_{a_{A,\Lambda}} = \{a_1t_1 + ... + a_nt_n : t = (t_1, t_2, ..., t_n) \in \Delta(\Lambda)\}. \tag{3.1}$$

It is instructive to understand the geometric regularities of its behaviour.

We start with the simplest case of just two limit values. Let $A = (a_1, a_2)$ and $\Lambda = \{\theta_1\}$. In this case the spectrum $\operatorname{sp} T_{a_{A,\Lambda}}$ *does not depend* on Λ, is *uniquely* determined by A, and is the straight line segment connecting the points a_1 and a_2. This is an effect of low dimension: *each curve connecting the vertices of a one-dimensional simplex is the simplex itself, and is the straight line segment connecting the vertices.*

Passing to $n > 2$ we consider first the most transparent case $n = 3$. In this case the curve $\Delta(\Lambda)$ lies on a two-dimensional simplex, which has the same dimension as the complex plane where the spectrum lies.

As we already know (see Figure 1), the continuous curve $\Delta(\Lambda)$ connecting the vertices $v_1 = (1, 0, 0)$ and $v_3 = (0, 0, 1)$ *does depend* essentially on Λ. Then by (3.1), the spectrum $\operatorname{sp} T_{a_{A,\Lambda}}$, geometrically, is the image of the curve $\Delta(\Lambda)$ under the projection of the two-dimensional simplex to the complex plane such that each its vertex v_k is projected to a_k, $k = 1, 2, 3$, and $a_k \in A$. That is, the set A determines the triangle to which the simplex is projected, while the set Λ determines the shape of the curve $\Delta(\Lambda)$, whose projection into the already defined triangle gives the spectrum.

In the next two pictures we illustrate this for three different sets Λ, being the first, third, and fifth set of angles of Figure 1. That is, we consider the following sets of angles $(0.48\pi, 0.52\pi)$, $(0.3\pi, 0.7\pi)$, and $(0.1\pi, 0.9\pi)$, ordered as generated from less to more curved lines. For the first picture the set A is given by $(0.1 + 0.1i, 0.9i, 0.9 + 0.5i)$, while $A = (0.1 + 0.1i, 1 + 0.2i, 0.9 + 0.5i)$, for the second picture. For both sets we leave the same values of a_1 and a_3, making the pictures "one-parametric" in dependence on a_2.

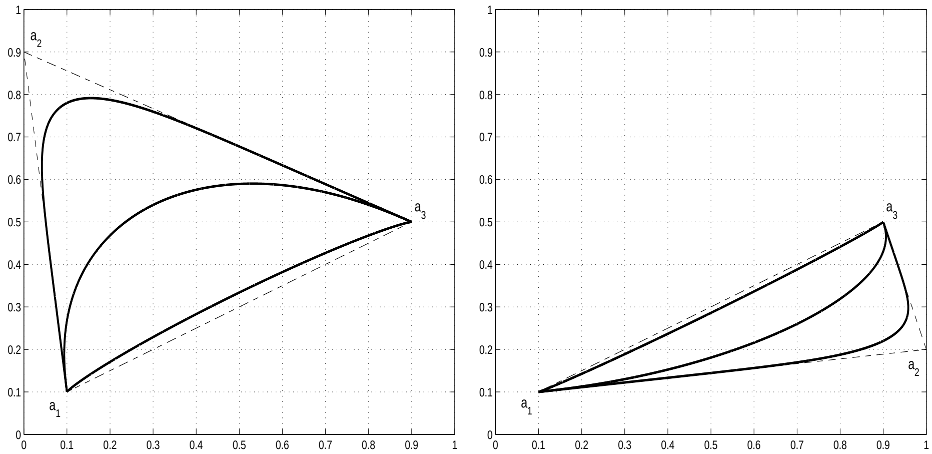

FIGURE 2. Spectra of $T_{a_{A,\Lambda}}$ for three limit values symbols.

The case $n > 3$ maintains in principle the same features. The spectrum $\operatorname{sp} T_{a_{A,\Lambda}}$ is the image of the curve $\Delta(\Lambda)$ under the projection of, now, the $(n-1)$-dimensional simplex onto a certain convex polygon in the complex plane such that each vertex v_k is projected to a_k, $k = 1, 2, , ..., n$, and $a_k \in A$. The curve $\Delta(\Lambda)$ connecting the vertices $v_1 = (1, 0, ..., 0)$ and $v_n = (0, ..., 0, 1)$ again *does depend* essentially on Λ. The set A determines the polygon to which the simplex is projected, while the set Λ determines the shape of the curve $\Delta(\Lambda)$, whose projection into the already defined polygon gives the spectrum. The only difference is that now this convex polygon has n or *less* vertices, depending on the way, prescribed by A, in which the $(n-1)$-dimensional simplex is projected onto the two-dimensional polygon. That is, the projections of some vertices may (or may not) be in the interior of the polygon.

In the next two pictures we present the cases of five limit values symbols for which the 4-dimensional simplex is projected onto a pentagon and a triangle, respectively. We consider the following sets A

$$(0.2 + 0.1i, 0.4 + 0.9i, 0.8 + 0.1i, 0.1 + 0.7i, 0.9 + 0.8i)$$

and

$$(0.2 + 0.1i, 0.5 + 0.6i, 0.1 + 0.9i, 0.3 + 0.4i, 0.9 + 0.8i),$$

maintaining the same values of a_1 and a_5 for both cases. Both pictures represent three spectra for the following sets Λ

$$(0.46\pi, 0.48\pi, 0.52\pi, 0.54\pi),\ (0.2\pi, 0.2\pi, 0.7\pi, 0.8\pi),\ (0.0002\pi, 0.01\pi, 0.99\pi, 0.9998\pi),$$

and which again correspond to lines ordered from less to more curved.

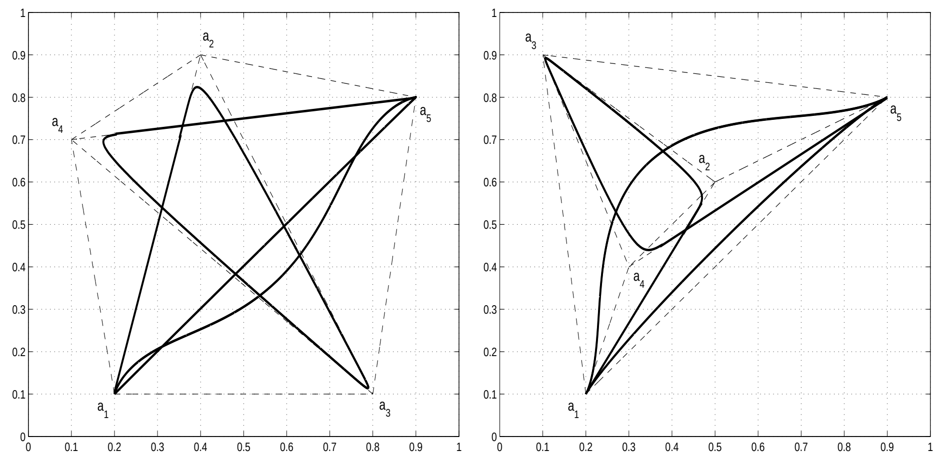

FIGURE 3. Spectra of $T_{a_{A,\Lambda}}$ for five limit values symbols (pentagon and triangle).

We note that the spectrum $\operatorname{sp} T_{a_{A,\Lambda}}$ becomes more rectilinear and more stable under the perturbations of $a_k \in A$, $k = 2, ..., n-1$, for bigger values of the angles θ_1 and $\pi - \theta_{n-1}$. In this case the spectrum approaches the straight line segment connecting the images of the vertices $(1, 0, ..., 0)$ and $0, ..., 0, 1)$ when the sum of these angles tends to π. The opposite, in a sense, tendency appears when the angles between the curves intersecting at t_0 and the boundary of the domain tend to 0. In that case the spectrum approaches the union of straight line segments passing in sequence through the images of the vertices $(1, 0, ..., 0)$, $(0, 1, 0, ..., 0)$, ... $(0, ..., 0, 1)$.

4. Local analysis at a point of discontinuity

Although the description given by Theorem 2.6 proves to be useful, it hides, at the same time, some essential properties of the above Toeplitz operator algebras. In particular, it turns out that each Toeplitz operator algebra $\mathcal{T}(PC(\overline{\mathbb{D}}, \ell))$, besides the initial generators T_a with symbols $a \in PC(\overline{\mathbb{D}}, \ell)$, contains many another Toeplitz operators with much more general symbols.

We show this here for the model situation at a point of discontinuity. We introduce first a number of symbol sets. Denote by $L_\infty^{\{0,\pi\}}(0, \pi)$ the C^*-subalgebra of $L_\infty(0, \pi)$ which consists of all functions having limits at the points 0 and π. Let $C[0, \pi]$ be, as usual, the algebra of all continuous functions on $[0, \pi]$; denote by $PC([0, \pi], \Lambda)$, where $\Lambda = \{\theta_1, \theta_2, ..., \theta_{n-1}\}$, the algebra of all piece-wise continuous functions on $[0, \pi]$, continuous in $[0, \pi] \setminus \Lambda$ and having one-sided limit values at the points of Λ. Let $PCo([0, \pi]), \Lambda)$ be the subalgebra of $PC([0, \pi], \Lambda)$ consisting of all piece-wise constant functions. Given a function $a_0(\theta)$, denote by $L(1, a_0)$ the linear two-dimensional space, generated by 1 and the function a_0.

Note that $PCo([0,\pi],\{\theta_1\}) = L(1,\chi_{[0,\theta_1]})$, where $\chi_{[0,\theta_1]}(\theta)$ is the characteristic function of $[0,\theta_1]$.

For a continuous function a_0, a set Λ, and an arbitrary point $\theta_k \in \Lambda$, we have the following chain of proper inclusions

$$\begin{array}{c} L(1,a_0) \subset C[0,\pi] \\ PCo([0,\pi],\{\theta_k\}) \subset PCo([0,\pi]),\Lambda) \end{array} \subset PC([0,\pi],\Lambda) \subset L_\infty^{\{0,\pi\}}(0,\pi). \tag{4.1}$$

Given a linear set $\mathcal{A}$, the subset of $L_\infty(0,\pi)$, denote by $H(\mathcal{A})$ the subset of $\mathcal{A}_\infty$ which consists of all homogeneous functions of zero order on the upper half-plane whose restrictions onto the upper half of the unit circle (parameterized by $\theta \in [0,\pi]$) belong to $\mathcal{A}$. Further let $\mathcal{T}(H(\mathcal{A}))$ be the the C^*-algebra generated by all Toeplitz operators T_a with symbols $a \in H(\mathcal{A})$.

Note that for any real nonconstant function a_0, the algebra $\mathcal{T}(H(L(1,a_0)))$ is a C^*-algebra with identity generated by a *single* self-adjoint element, the Toeplitz operator T_{a_0}.

Let $\mathcal{A}$ be any of the sets in (4.1), consider the C^*-algebra $\mathcal{T}(H(\mathcal{A}))$.

For the largest set (algebra) $L_\infty^{\{0,\pi\}}(0,\pi)$ we have

Theorem 4.1. *The C^*-algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$ is isomorphic and isometric to $C(\overline{\mathbb{R}})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is the two-point compactification of $\mathbb{R}$. The isomorphic isomorphism*

$$\tau_\infty : \mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi))) \longrightarrow C(\overline{\mathbb{R}})$$

is generated by the following mapping of generators of the algebra $\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))$

$$\tau_\infty : T_a \longmapsto \gamma_a(\lambda), \tag{4.2}$$

where $a = a(\theta) \in H(L_\infty^{\{0,\pi\}}(0,\pi))$.

Proof. We need to show only that the mapping (4.2) is onto. The inclusion

$$\tau_\infty(\,\mathcal{T}(H(L_\infty^{\{0,\pi\}}(0,\pi)))\,) \subset C(\overline{\mathbb{R}})$$

is trivial. The inverse inclusion will follow from the next theorem. □

Passing to another extreme, the smallest possible set, we have

Theorem 4.2. *Let $a_0(\theta) \in L_\infty^{\{0,\pi\}}(0,\pi)$ be a real valued function such that the function $\gamma_{a_0}(\lambda)$ separates the points of $\overline{\mathbb{R}}$. Then the C^*-algebra $\mathcal{T}(H(L(1,a_0)))$ is isomorphic and isometric to $C(\overline{\mathbb{R}})$. The isomorphic isomorphism*

$$\tau_\infty : \mathcal{T}(H(L(1,a_0))) \longrightarrow C(\overline{\mathbb{R}})$$

is generated by the same mapping of generators of the algebra $\mathcal{T}(H(L(1,a_0)))$

$$\tau_\infty : T_a \longmapsto \gamma_a(\lambda),$$

Proof. Follows directly from the Stone-Weierstrass theorem. □

Corollary 4.3. *Given a point $\theta_0 \in (0,\pi)$, the C^*-algebra $\mathcal{T}(H(PCo([0,\pi],\{\theta_0\})))$ is isomorphic and isometric to $C(\overline{\mathbb{R}})$.*

Proof. As it was already mentioned $PCo([0,\pi],\{\theta_0\}) = L(1,\chi_{[0,\theta_0]})$. All we need to prove is that the real valued function

$$\gamma_{\chi_{[0,\theta_0]}}(\lambda) = \frac{2\lambda}{1-e^{-2\pi\lambda}}\int_0^{\theta_0} e^{-2\lambda\theta}\,d\theta = \frac{e^{-2\theta_0\lambda}-1}{e^{-2\pi\lambda}-1}$$

separates the points of $\overline{\mathbb{R}}$. We show that the function $\gamma_{\chi_{[0,\theta_0]}}$ is strictly increasing by a simple but somewhat lengthy procedure.

After the scaling $t = 2\pi\lambda$, $\theta_0 = \alpha\pi$, with $\alpha \in (0,1)$, we have

$$\gamma(t) = \frac{e^{-\alpha t}-1}{e^{-t}-1}, \qquad t \in \mathbb{R}.$$

First let $t > 0$, and calculate

$$\gamma'(t) = \frac{\alpha e^{-\alpha t}(1-e^{-t}) - e^{-t}(1-e^{-\alpha t})}{(1-e^{-t})^2}.$$

To show that $\gamma'(t) > 0$, it is equivalent to show that

$$\alpha e^{-\alpha t}\frac{e^t-1}{e^t} - e^{-t}\frac{e^t-1}{e^{\alpha t}} > 0,$$

or that

$$\alpha(e^t-1) - (e^{\alpha t}-1) > 0$$

or

$$\sum_{k=1}^{\infty}(\alpha-\alpha^k)\frac{t^k}{k!} > 0.$$

The last inequality is evident because $\alpha \in (0,1)$.

Pass now to $t < 0$. Substituting $x = -t$, $x \in \mathbb{R}_+$, we have

$$\gamma(t(x)) = \frac{e^{\alpha x}-1}{e^x-1}$$

and

$$\gamma'(t(x)) = \frac{\alpha e^{\alpha x}(e^x-1) - e^x(e^{\alpha x}-1)}{(e^x-1)^2}.$$

Now we need to show that the function $\gamma(t(x))$ is strictly decreasing, or that $\gamma'(t(x)) < 0$. This is equivalent to

$$e^x(e^{\alpha x}-1) - \alpha e^{\alpha x}(e^x-1) > 0$$

or to

$$(1-\alpha)(e^x-1) - (e^{(1-\alpha)x}-1) > 0,$$

or to

$$\sum_{k=1}^{\infty}[(1-\alpha)-(1-\alpha)^k]\frac{x^k}{k!} > 0.$$

Again the last inequality is evident because $\alpha \in (0,1)$. □

Let now $a_0(\theta) = \frac{\theta}{\pi}$, this function $a_0(\theta)$ is obviously real valued and continuous on $[0,\pi]$.

Corollary 4.4. *The C^*-algebra $\mathcal{T}(H(L(1,a_0)))$ is isomorphic and isometric to $C(\overline{\mathbb{R}})$.*

Proof. We have

$$\begin{aligned}\gamma_{a_0}(\lambda) &= \frac{2\lambda}{\pi(1-e^{-2\pi\lambda})}\int_0^\pi \theta e^{-2\lambda\theta}\,d\theta\\ &= \frac{1}{\pi(1-e^{-2\pi\lambda})}\left[-\pi e^{-2\pi\lambda}-\frac{1}{2\lambda}(e^{-2\pi\lambda}-1)\right]=\frac{1}{2\pi\lambda}-\frac{1}{e^{2\pi\lambda}-1}.\end{aligned}$$

The function $\gamma_{a_0}(\lambda)$ is continuous on $\overline{\mathbb{R}}$, and

$$\begin{aligned}\lim_{\lambda\to 0}\gamma_{a_0}(\lambda) &= \gamma_{a_0}(0)=\frac{1}{2}\\ \lim_{\lambda\to -\infty}\gamma_{a_0}(\lambda) &= \gamma_{a_0}(-\infty)=1\\ \lim_{\lambda\to +\infty}\gamma_{a_0}(\lambda) &= \gamma_{a_0}(+\infty)=0.\end{aligned}$$

To finish the proof we need to show that the function $\gamma_{a_0}(\lambda)$ separates the points of $\overline{\mathbb{R}}$. To do this we show that the function

$$\gamma(t)=\frac{1}{t}-\frac{1}{e^t-1}, \qquad t\in\mathbb{R}$$

is strictly decreasing, or that $\gamma'(t)<0$ for all $t\neq 0$.

The function

$$\gamma'(t)=-\frac{1}{t^2}+\frac{e^t}{(e^t-1)^2}$$

is even. Thus it is sufficient to prove that

$$\frac{e^t}{(e^t-1)^2}<\frac{1}{t^2},$$

or that

$$t^2e^t<(e^t-1)^2,$$

for each $t>0$. The last inequality is easy to check, comparing coefficients of the power series

$$\begin{aligned}t^2e^t &= \sum_{n=2}^\infty \frac{1}{(n-2)!}\,t^n,\\ (e^t-1)^2 &= \sum_{n=2}^\infty \frac{2(2^{n-1}-1)}{n!}\,t^n.\end{aligned}$$

□

Remark 4.5. The above statements show that in spite of the fact that the generating sets of symbols in (4.1) are quite different, the resulting Toeplitz C^*-algebras are the same. Moreover, this (common) C^*-algebra with identity can be generated by a *single* Toeplitz operator with either *continuous*, or *piece-wise constant* symbol. Further, although the algebraic operations with Toeplitz operators do not give a

Toeplitz operator, in general, the resulting (single-generated) algebra is extremely rich in Toeplitz operators: *each Toeplitz operator with symbol from* $H(L_\infty^{\{0,\pi\}}(0,\pi))$ *belongs to this algebra.*

We give now a number of illustrating examples. Consider $\mathcal{A}^2(\Pi)$ and the Toeplitz operator T_+ with symbol $a_+(z) = \chi_+(\operatorname{Re} z) = \chi_+(x)$, where χ_+ is the characteristic function of the positive half-line. We have as well that $a_+(z) = a_+(re^{i\theta}) = \chi_{[0,\pi/2]}(\theta)$, and thus $a_+ \in H(PCo([0,\pi],\{\pi/2\}))$.

The Toeplitz operator $T_+ \in \mathcal{T}(H(PCo([0,\pi],\{\pi/2\})))$ is unitary equivalent to the multiplication operator $\gamma_{a_+}I$, where, by (2.2),

$$\gamma_{a_+}(\lambda) = \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^\pi \chi_{[0,\pi/2]}(\theta)\, e^{-2\lambda\theta}\, d\theta = \frac{e^{-\pi\lambda}-1}{e^{-2\pi\lambda}-1}, \qquad \lambda \in \mathbb{R}.$$

The operator T_+ is obviously self-adjoint and $\operatorname{sp} T_+ = [0,1]$. Thus for any function f continuous on $[0,1]$ the operator $f(T_+)$ is well defined by the standard functional calculus in C^*-algebras, furthermore the operator $f(T_+)$ belongs to the same algebra $\mathcal{T}(H(PCo([0,\pi],\{\pi/2\})))$.

Example. Consider the family of functions f_α parameterized by $\alpha \in [0,1]$ and given as follows:

$$f_\alpha(x) = x^{2(1-\alpha)}\, \frac{(1-x)^{2\alpha} - x^{2\alpha}}{(1-x)-x}, \qquad x \in [0,1]. \tag{4.3}$$

Each function f_α is continuous on $[0,1]$, and $f_\alpha(0) = 0$, $f_\alpha(1) = 1$. Let us mention as well some particular cases

$$f_0(x) \equiv 0, \qquad f_{\frac{1}{2}}(x) = x, \qquad f_1(x) \equiv 1.$$

Then

$$f_\alpha(T_+) = T_{\chi_{[0,\alpha\pi]}} \in \mathcal{T}(H(PCo([0,\pi],\{\pi/2\}))),$$

where the symbol $\chi_{[0,\alpha\pi]}$ of the operator $T_{\chi_{[0,\alpha\pi]}}$ belongs to $H(PCo([0,\pi],\{\alpha\pi\}))$.

Proof. We will exploit the isomorphism between the Toeplitz operator algebra and the functional algebra given in Corollary 2.4. Introduce

$$x = \gamma_{a_+}(\lambda) = \frac{e^{-\pi\lambda}-1}{e^{-2\pi\lambda}-1} = \frac{1}{e^{-\pi\lambda}+1} \in [0,1],$$

which is equivalent to

$$\lambda = \lambda(x) = -\frac{1}{\pi}\, \ln \frac{1-x}{x}.$$

Then for the operator $T_{\chi_{[0,\alpha\pi]}}$ the corresponding function $\gamma_{\chi_{[0,\alpha\pi]}}$ is given by

$$\gamma_{\chi_{[0,\alpha\pi]}}(\lambda) = \frac{2\lambda}{1-e^{-2\pi\lambda}} \int_0^\pi \chi_{[0,\lambda\pi]}(\theta)\, e^{-2\lambda\theta}\, d\theta = \frac{e^{-2\alpha\pi\lambda}-1}{e^{-2\pi\lambda}-1}, \qquad \lambda \in \mathbb{R}.$$

Substituting $\lambda = \lambda(x)$ we have

$$\begin{aligned}\gamma_{\chi_{[0,\alpha\pi]}}(\lambda(x)) &= \frac{e^{2\alpha\pi\frac{1}{\pi}\ \ln\frac{1-x}{x}} - 1}{e^{2\pi\frac{1}{\pi}\ \ln\frac{1-x}{x}} - 1} \\ &= \frac{\left(\frac{1-x}{x}\right)^{2\alpha} - 1}{\left(\frac{1-x}{x}\right)^{2} - 1} = x^{2(1-\alpha)}\,\frac{(1-x)^{2\alpha} - x^{2\alpha}}{(1-x) - x}.\end{aligned}$$

□

Note that the above mentioned particular cases of f_α lead to the equalities

$$f_0(T_+) = 0, \qquad f_{\frac{1}{2}}(T_+) = T_+, \qquad f_1(T_+) = I,$$

as it should be.

In the next example we present a connection between Toeplitz operators with piece-wise constant symbols having just two and more than two limit values at the single point of discontinuity.

Example. Given a finite ordered set of numbers $0 < \alpha_1 < \alpha_2 < ... < \alpha_{n-1} < 1$, introduce

$$\Lambda = \{\alpha_1\pi, \alpha_2\pi, ..., \alpha_{n-1}\pi\};$$

for convenience we add $\alpha_0 = 0$ and $\alpha_n = 1$. Let further $A = \{a_1, a_2, ..., a_n\}$ be an ordered set of complex numbers.

Given both A and Λ, we define the piece-wise constant symbol

$$a_{A,\Lambda}(\theta) = \sum_{k=1}^{n} a_k \chi_{(\alpha_{k-1}\pi, \alpha_k\pi]} \in PCo([0,\pi], \Lambda)$$

and the function $f_{A,\Lambda} = f_{A,\Lambda}(x)$ continuous on $[0, 1]$

$$f_{A,\Lambda}(x) = \sum_{k=1}^{n} a_k\,\frac{(1-x)^{2\alpha_k}x^{2(1-\alpha_k)} - (1-x)^{2\alpha_{k-1}}x^{2(1-\alpha_{k-1})}}{(1-x) - x}.$$

Then

$$f_{A,\Lambda}(T_+) = T_{a_{A,\Lambda}} \in \mathcal{T}(H(PCo([0,\pi], \{\pi/2\}))).$$

Proof. Consider the Toeplitz operator $T_{a_{A,\Lambda}}$. Using (2.3) we have

$$\begin{aligned}\gamma_{a_{A,\Lambda}}(\lambda) &= \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_0^\pi a_{A,\Lambda}(\theta)\, e^{-2\lambda\theta}\, d\theta \\ &= \sum_{k=1}^{n} a_k\,\frac{e^{-2\alpha_k\pi\lambda} - e^{-2\alpha_{k-1}\pi\lambda}}{e^{-2\pi\lambda} - 1}, \qquad \lambda \in \mathbb{R}.\end{aligned}$$

Substitute $\lambda = \lambda(x)$ as in the previous example. Then after a simple calculation we have

$$\gamma_{a_{A,\Lambda}}(\lambda(x)) = \sum_{k=1}^{n} a_k\,\frac{(1-x)^{2\alpha_k}x^{2(1-\alpha_k)} - (1-x)^{2\alpha_{k-1}}x^{2(1-\alpha_{k-1})}}{(1-x) - x}.$$

□

Theorem 4.2 and Corollary 4.3 imply, in particular, that each Toeplitz operator with $H(L_\infty^{\{0,\pi\}}(0,\pi))$-symbol can be obtained in a similar way. The exact formula for the corresponding continuous function $f(x)$, though forcedly rather implicit, is given in the next example.

Example. Given a function $a = a(\theta) \in H(L_\infty^{\{0,\pi\}}(0,\pi))$, let

$$f_a(x) = \frac{2x^2}{\pi}\,\frac{\ln(1-x) - \ln x}{(1-x) - x} \int_0^\pi a(\theta) \left(\frac{1-x}{x}\right)^{\frac{2\theta}{\pi}} d\theta.$$

Then

$$f_a(T_+) = T_a.$$

Remark 4.6. In the above examples we have considered the Toeplitz operator T_+ as the starting operator by a very simple reason: in this specific case the generically transcendental equation $x = \gamma_a(\lambda)$ admits an explicit solution.

We can start as well from any Toeplitz operator T_α having the symbol $\chi_{[0,\alpha\pi]}$, where $\alpha \in (0,\pi)$. Indeed, as follows from the proof of Corollary 4.3, the function $\gamma_{\chi_{[0,\alpha\pi]}}(\lambda)$ is strictly increasing. This implies that the function $f_\alpha(x)$ (see (4.3)), which maps $[0,1]$ onto $[0,1]$, is strictly increasing as well. Thus the function $f_\alpha^{-1}(x)$ is well defined and continuous on $[0,1]$.

Finally, given $\alpha, \beta \in (0,\pi)$, A, Λ, and $a = a(\theta) \in H(L_\infty^{\{0,\pi\}}(0,\pi))$, we have

$$T_\alpha \in \mathcal{T}(H(PCo([0,\pi],\{\alpha\pi\})))$$

and

$$\begin{array}{rcl} (f_\beta \circ f_\alpha^{-1})(T_\alpha) & = & T_\beta, \\ (f_{A,\Lambda} \circ f_\alpha^{-1})(T_\alpha) & = & T_{a_{A,\Lambda}}, \\ (f_a \circ f_\alpha^{-1})(T_\alpha) & = & T_a, \end{array}$$

where all Toeplitz operators from the right hand side of the above equalities belong to $\mathcal{T}(H(PCo([0,\pi],\{\alpha\pi\})))$.

5. Boundary piecewise continuous functions

The above examples show that studying the algebra generated by Toeplitz operators, whose symbols admit discontinuities at a finite number of boundary points, we can start from any symbol algebra selected from a wide variety of symbol classes. Moreover, the curve ℓ, entering in the definition of the symbol algebra $PC(\overline{\mathbb{D}},\ell)$, does not play in fact any significant role. In all such cases the resulting C^*-algebra will contain all Toeplitz operators whose symbols admit a "homogeneous type discontinuity" in each boundary point of discontinuity, locally described by the algebra $H(L_\infty^{\{0,\pi\}}(0,\pi))$.

Thus it seems reasonable to include the Toeplitz operators with such symbols among the generators of the algebra from the very beginning. In this case the definition proceeds as follows.

Let $T = \{t_1, t_2, ..., t_m\}$ be a finite set of distinct points on the unit circle $\gamma = \partial\mathbb{D}$. Introduce the linear space $BPC(\mathbb{D}, T)$ (BPC stands for *Boundary Piecewise Continuous*) which consists of all functions $a(z)$ obeying the following properties:

(i) $a(z) \in L_\infty(\mathbb{D})$;

(ii) $a(z)$ has limit values at all boundary point $t \in \gamma \setminus T$, and the function $a(t)$ constructed by these limit values is continuous in $\gamma \setminus T$;

(iii) at each point $t_0 \in T$ the function $a(z)$ has a "homogeneous type discontinuity", which means that there exist a Möbius transformation $z = z_{t_0}(w)$ of the upper half-plane Π to the unit disk $\mathbb{D}$ with $t_0 = z_{t_0}(0)$ and a homogeneous function of order zero $a_{t_0}(w) \in H(L_\infty^{\{0,\pi\}}(0, \pi))$ such that

$$\lim_{w \to 0} [a(z_{t_0}(w)) - a_{t_0}(w)] = 0.$$

Let us make several comments on this definition. The set $BPC(\mathbb{D}, T)$ in fact is a C^*-algebra, although only the linear space structure is important for our purposes. The function $a(t)$, as a function of the boundary points, belongs to $PC(\gamma, T)$; that is, for each point $t_0 \in T$ the following limits

$$\lim_{t \to t_0,\, t \prec t_0} a(t) = a(t_0 - 0) \qquad \text{and} \qquad \lim_{t \to t_0,\, t_0 \prec t} a(t) = a(t_0 + 0)$$

are well defined. Property (iii) of the above definition can be alternatively done in geometric terms of $\mathbb{D}$ as follows. For each point $t_0 \in T$ there are a hyperbolic pencil $\mathcal{P}_{t_0}$ of geodesics in $\mathbb{D}$, such that t_0 is the endpoint of its axis, and a function $\widetilde{a}_{t_0}(z)$ which is constant on cycles of $\mathcal{P}_{t_0}$ and whose values on (each) geodesic are given by an L_∞-function having limit values at the endpoints of the geodesic on γ (points at infinity in hyperbolic geometry), such that

$$\lim_{z \to t_0} [a(z) - \widetilde{a}_{t_0}(z)] = 0.$$

Consider now the C^*-algebra $\mathcal{T}_{BPC} = \mathcal{T}(BPC(\mathbb{D}, T))$ generated by all Toeplitz operators T_a with symbols $a \in BPC(\mathbb{D}, T)$.

Let, as above, $\widehat{\gamma}$ be the set γ, cut by points $t_p \in T$. The pair of points which correspond to a point $t_p \in T$, $p = \overline{1, m}$, we denote by $t_p - 0$ and $t_p + 0$, following the positive orientation of γ. Let $\overline{X} = \sqcup_{p=1}^m \Delta_p$ be the disjoint union of segments $\Delta_p = [0, 1]$. Denote by Γ the union $\widehat{\gamma} \cup \overline{X}$ with the following point identification

$$t_p - 0 \equiv 1_p, \qquad t_p + 0 \equiv 0_p,$$

where $t_p \pm 0 \in \widehat{\gamma}$, 0_p and 1_p are the endpoints of Δ_p, $p = 1, 2, ..., m$.

Than we have obviously

Theorem 5.1. *The C^*-algebra $\mathcal{T}_{BPC} = \mathcal{T}(BPC(\mathbb{D}, T))$ is irreducible and contains the ideal $\mathcal{K}$ of compact operators. The symbol algebra* $\operatorname{Sym} \mathcal{T}_{BPC} = \mathcal{T}_{BPC}/\mathcal{K}$ *is* *isomorphic to the algebra* $C(\Gamma)$*. Identifying them, the symbol homomorphism*

$$\mathrm{sym}\,: \mathcal{T}_{BPC} \to \mathrm{Sym}\,\mathcal{T}_{BPC} = C(\Gamma)$$

is generated by the following mapping of generators of $\mathcal{T}_{BPC}$

$$\mathrm{sym}\,: T_a \longmapsto \begin{cases} a(t), & t \in \widehat{\gamma} \\ \gamma_{a_{t_p}}\left(\frac{1-2x}{\sqrt{1-(1-2x)^2}}\right), & x \in [0,1] \end{cases},$$

where a_{t_p} *is the function defined by the above property (iii) for* $a(z)$ *at the point* $t_p \in \Lambda$*,* $p = 1, 2, ..., m$*, and*

$$\gamma_{a_{t_p}}(\lambda) = \frac{2\lambda}{1 - e^{-2\pi\lambda}} \int_0^\pi a_{t_p}(\theta)\, e^{-2\lambda\theta}\, d\theta, \qquad \lambda \in \mathbb{R}.$$

An operator $T \in \mathcal{T}_{BPC}$ *is Fredholm if and only if its symbol is invertible, i.e., the function* $\mathrm{sym}\, T \neq 0$ *on* Γ*, and*

$$\mathrm{Ind}\, T = -\frac{1}{2\pi}\{\mathrm{sym}\, T\}_\Gamma.$$

Proof. Easily follows from the standard local principle, Theorem 4.1 and Theorem 2.6. □

We mention that the algebras described by Theorems 2.2, 2.6, and 5.1 consist of the same operators, in spite of the fact that their initial generators are quite different. That is, as it turned out, the first algebra generated by Toeplitz operators with discontinuous symbols, which was described by Theorem 2.2, already contained all the operators with $BPC(\mathbb{D}, T)$-symbols. For about twenty years there was no way to see this. At the same time Theorem 5.1 gives a transparent description for all Toeplitz operators for all $BPC(\mathbb{D}, T)$-symbols.

We end the paper formulating two open problems.

Problem 1. Extend the description of Toeplitz operator algebra from $BPC(\mathbb{D}, T)$-symbols to a rotation invariant symbol set containing $BPC(\mathbb{D}, T)$.

Problem 2. Extend the description of Toeplitz operator algebra from $BPC(\mathbb{D}, T)$-symbols to a Möbius invariant symbol set containing $BPC(\mathbb{D}, T)$. This class of symbols can be naturally called $BPC(\mathbb{D})$.

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